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Transverse instability of plane wavetrains in gas-fluidized beds

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Using a two-fluid model of gas-fluidized beds, it is shown that periodic plane voidage waves travelling against gravity are unstable to perturbations with large transverse wavelength. This secondary instability sets in at arbitrarily small amplitudes of the plane wave and correspondingly small transverse wavenumbers of the two-dimensional perturbation. More precisely, if the bed is wide enough to accommodate sufficiently long horizontal waves, then the plane wave becomes unstable as soon as its amplitude has grown to the order of the square of the transverse wavenumber. The instability can be stationary or oscillatory in nature and has its origin in the interaction between the plane wave and four least-stable modes with small transverse wavenumber. Two of them represent a pair of bubble-like 'mixed modes'; the other two are initially, i.e. at the onset of the primary wave, pure transverse modes, one consisting only of an initially pure vertical velocity perturbation of the state of uniform fluidization. Depending on a relation between the eigenvalues of the least-stable modes at the primary bifurcation point, either one of these can be the dominant mode, which becomes (most) unstable along the growing vertically travelling plane wave. While the transverse modes gain longitudinal structure during this process, the mixed modes obtain a vertical component of the vertically averaged velocity as well, so that it appears that the secondary instability described here is a variant of Batchelor & Nitsche's (1991) 'overturning' instability found recently for unbounded stratified fluids, see also Batchelor (1993).

1. Introduction

The search for and speculation about the origin of bubbles in gas-fluidized beds has a long tradition (see e.g. El-Kaissy & Homsy 1976; Didwania & Homsy 1982; Needham & Merkin 1984*a*; Batchelor 1993). As the main instability of the state of uniform fluidization is in the vertical direction, giving rise to upwards travelling plane waves, it is commonly believed that bubbles appear via a secondary instability due to transverse perturbations of these waves.

In what follows, we shall identify such an instability mechanism by tracking the eigenvalues of the least-stable modes from the primary bifurcation point to small but finite amplitudes of the plane vertically travelling periodic wave (VTW). By this we mean the following: at the primary bifurcation point the largest eigenvalue is zero, signalling the onset of the one-dimensional VTW; this eigenvalue is real and counts doubly, since we shall be working in a frame moving with the VTW (which affects only the imaginary parts of the eigenvalues, and transforms the imaginary parts of the critical eigenvalues into zero). Among the infinitely many eigenvalues with negative real part there are four (semi-simple, double-counting due to a horizontal-reflectional symmetry), which are proportional to $-k^2$, k being the transverse wavenumber, so that they come close to criticality when k becomes small. The corresponding eigenfunctions consist of two pure transverse modes with real eigenvalues, and a pair of two-dimensional ('mixed') modes, periodic in each direction, with complex-conjugate eigenvalues.

These modes are present in the linear stability problem of the basic state of uniform fluidization, but did not receive much attention from previous investigators, presumably because they are not responsible for the primary instability. This applies particularly to the two transverse modes, which are stable perturbations of the uniform state for all wavenumbers, one having the additional 'disadvantage' of representing a pure velocity perturbation, while the main interest was in the search for voidage non-uniformities. However, the transverse modes are necessary to enforce the mixed modes by interaction with the primary plane wave. Of course, the opposite interaction of the mixed modes with the plane wave also enhances the transverse modes, and it depends on the relative magnitude of the eigenvalues of the two transverse modes compared to those of the mixed modes at the starting (primary bifurcation) point, which type of mode wins the competition and is most responsible for the occurrence of a secondary instability.

The relevance of the mixed modes has been demonstrated previously by the author as these become unstable along the uniform state as well, thereby giving rise to two-dimensional periodic travelling waves (Göz 1992). Because their bifurcation point depends on the transverse wavenumber k, and lies close to the bifurcation point of the VTW for small k, it appears natural that these modes also play an important role in the instability of the VTW. The significance of the pure transverse perturbations of the uniform state for the nonlinear behaviour of fluidized beds, however, is a completely novel perception.

The revealed instability mechanism differs from the idea of Didwania & Homsy (1982) concerning a resonant sideband instability, in that it is less the second harmonic of the primary wave but mainly the modes with only a transverse structure which feed energy into the mixed modes by interacting with the planar wavetrain, and vice versa.

While the transverse modes exist for non-vanishing transverse wavenumbers k only, the considerations mentioned above suggest that the most 'dangerous' regime will be that of small k. We shall indeed prove that the secondary instability becomes effective for VTW amplitudes which are as small as k^2, and that then the eigenvalues of the disturbance packet are of the same scale. At this stage all the perturbation modes have grown into bubble-like mixed modes, with an additional vertical velocity component that varies sinusoidally in the horizontal direction. The secondary instability found here is therefore similar to the 'overturning' instability for unbounded stratified fluids (Batchelor & Nitsche 1991; see also Batchelor 1993), although the details are even more involved.

We consider the reduced model applicable to gas-fluidized beds – and other single- and multi-phase flow systems (e.g. Whitham 1974; Needham & Merkin 1984*b*; Kerner & Konhäuser 1994; Lahey & Drew 1989; Doi & Onuki 1992; Göz 1994) – as described in Garg & Pritchett (1975) and Göz (1992), but now formulated in a moving coordinate system:

$$-\partial_t \phi + \nabla\cdot[(1-\phi)(\boldsymbol{v}-\omega \boldsymbol{k})] = 0, \tag{1.1a}$$

$$\nabla\cdot[\boldsymbol{v}-\varphi(\phi)\nabla p] = 0, \tag{1.1b}$$

$$F(1-\phi)[\partial_t \boldsymbol{v}+(\boldsymbol{v}-\omega\boldsymbol{k})\cdot\nabla\boldsymbol{v}] = -(1-\phi)\boldsymbol{k}-\nabla p-G(\phi)\nabla\phi+\mu(\Delta+\kappa\nabla\nabla\cdot)\boldsymbol{v}, \tag{1.1c}$$

$$B(\phi)(\boldsymbol{u}-\boldsymbol{v})+\phi\nabla p = 0. \tag{1.1d}$$

Here, $\nabla = (\partial_z, \partial_y)$ with $z = x-\omega t$ denoting the vertical coordinate, and ω being simultaneously the wave velocity and bifurcation parameter; $\boldsymbol{k}$ represents the unit vector against gravity, and y the horizontal direction. The dependent variables are gas volume fraction or voidage ϕ, particulate-phase velocity $\boldsymbol{v}$, and effective gas pressure p; the gas velocity $\boldsymbol{u}$ follows from the decoupled equation (1.1*d*).

The rescaling of spatial coordinates and velocities has been performed using the particle radius r and the minimum fluidization velocity u_0, respectively. This introduces the following parameters: Froude number $F = u_0^2/gr$, Reynolds number $R = \rho_s u_0 r/\mu_s$, and the viscosity coefficient $\kappa = (\lambda_s + \mu_s/3)/\mu_s$, where g denotes the gravity constant, ρ_s the specific density of the particles, and λ_s and μ_s the effective volume and shear viscosity coefficients of the particulate phase, which here we consider constant. It is also convenient to introduce the abbreviation $\mu = F/R$.

In addition, $G(\phi) < 0$ represents an elasticity modulus corresponding to interparticle forces, while the variable coefficient

$$\varphi(\phi) = \frac{\phi^2}{B(\phi)} = \frac{\phi^{n+2}}{(1-\phi)\,\phi_0^{n+1}} \tag{1.2}$$

is related to the drag force. Although not explicitly needed, the usual Stokes-like form of the drag coefficient $B(\phi)$ is employed, where n is the well-known Richardson–Zaki exponent describing empirically the expansion of a uniformly fluidized bed. This basic state is given by

$$\phi \equiv \phi_0, \quad \boldsymbol{v} = 0, \quad \boldsymbol{u} = \boldsymbol{u}_0 \equiv \boldsymbol{k}, \quad \nabla p = p_0' \boldsymbol{k}; \quad p_0' = -\frac{\phi_0}{\varphi_0} = -(1-\phi_0), \tag{1.3}$$

where the basic voidage value would be given by the relation $\phi_0^{n+1} = 9\mu_g \mu/[2\mu_s(1-\rho_g/\rho_s)]$. Note, however, that in deriving (1.1*d*) terms of the order of μ_g/μ_s and ρ_g/ρ_s have been omitted, so that this relation becomes meaningless, i.e. ϕ_0 has to be considered as being prescribed externally.

Needham & Merkin (1986) were the first to prove rigorously that the base state loses its stability to a one-dimensional periodic travelling wave. Recently an extensive treatment of vertical and oblique travelling plane waves for gas- and liquid-fluidized beds has been presented by Göz (1993*b*). The plane vertical travelling waves (VTW) in gas-fluidized beds are special stationary one-dimensional solutions of (1.1*a–c*), an approximation of which will be obtained in §2 by means of an amplitude expansion. In §3 the eigenvalue problem for (1.1) linearized at the VTW will be examined. There the perturbation variables and the eigenvalues are also expanded with respect to the amplitude of the VTW, while the transverse wavenumber of the perturbation is considered a fixed parameter. The zeroth-order approximation gives all modes together with their eigenvalues, which are present at the primary bifurcation point. It is found that there are four least-stable modes (modulo the reflectional symmetry in y), two belonging to a two-dimensional perturbation ('mixed modes') and two representing pure transverse perturbations. The latter are followed up to higher orders in the amplitude of the VTW; because this is very formal, only the results are described in §3.3, details can be found in the Appendix. The stability boundary cannot be reached in this way, but it turns out that the expansion breaks down for very small or very large transverse wavenumbers. Appropriate rescalings of the transverse variables with the expansion parameter and redesigned expansions, in which the transverse wavenumber is now allowed to vary with the amplitude of the VTW, reveal the presence and nature of a long-wave instability; this is carried out in §4. At the end of this study we shall describe the possible secondary bifurcating solutions using symmetry arguments and the knowledge gained about the critical eigenvalue(s).

2. Expansion of the one-dimensional solution

First, we must find an approximation to the periodic plane wave travelling against gravity. This solution is determined by the equations (see Göz 1993*a*, *b*)

$$(1-\phi)(v-\omega) = -\omega(1-\phi_0), \tag{2.1a}$$

$$v-\lambda\varphi\tilde{p}'-p_0'\varphi = -p_0'\varphi_0, \tag{2.1b}$$

$$\begin{aligned}\lambda F(1-\phi)(v-\omega)v' &= -F\lambda\omega(1-\phi_0)v' \\ &= \phi-\phi_0-\lambda\tilde{p}'-\lambda G(\phi)\phi'+\mu(1+\kappa)\lambda^2 v'',\end{aligned} \tag{2.1c}$$

with a prime denoting $\mathrm{d}/\mathrm{d}z$, $v \equiv v_z$ ($v_y \equiv 0$), $\varphi_0 = \varphi(\phi_0)$. Thereby, the unknown period of the solution has been scaled to 2π, such that the wavenumber λ occurs explicitly in the above equations; this has been achieved by the replacement $z \to z/\lambda$. Additionally, p_0' has been extracted from p' before scaling, since it is a constant:

$$\tilde{p}' = p'-p_0', \quad p_0' = -\frac{\phi_0}{\varphi_0} = -(1-\phi_0). \tag{2.2}$$

From (2.1) a second-order equation can be derived for ϕ alone:

$$c\lambda^2\omega\left(\phi''+2\frac{(\phi')^2}{1-\phi}\right)+\lambda\phi'\left[F\omega^2+\frac{(1-\phi)^2}{(1-\phi_0)^2}G(\phi)\right] \\ +\frac{1-\phi}{(1-\phi_0)^2\varphi(\phi)}\{-\omega(\phi-\phi_0)+(1-\phi)[(1-\phi)\varphi(\phi)-(1-\phi_0)\varphi_0]\} = 0, \tag{2.3}$$

with

$$c = \mu\frac{1+\kappa}{1-\phi_0}. \tag{2.4}$$

The other variables follow from

$$v = \omega\frac{\phi_0-\phi}{1-\phi}, \quad \lambda\tilde{p}' = \frac{1}{\varphi}\left[p_0'(\varphi_0-\varphi)+\omega\frac{\phi_0-\phi}{1-\phi}\right]. \tag{2.5}$$

The bifurcating periodic solutions can be determined by expansion in the parameter ϵ measuring the amplitude of the wave, in the following manner:

$$\left.\begin{aligned} &\omega = \omega_0+\epsilon^2\omega_2+\ldots, \quad \lambda = \lambda_0+\epsilon^2\lambda_2+\ldots, \\ &\tilde{\phi} = \phi_0+\epsilon\phi^0(\epsilon), \quad \phi^0(\epsilon) = \phi_1+\epsilon\phi_2+\epsilon^2\phi_3+\ldots, \\ &\tilde{v} = \epsilon v^0(\epsilon), \quad v^0(\epsilon) = v_1+\epsilon v_2+\epsilon^2 v_3+\ldots, \\ &\tilde{p}' = \epsilon p^{0\prime}(\epsilon), \quad p^{0\prime}(\epsilon) = p_1'+\epsilon p_2'+\epsilon^2 p_3'+\ldots. \end{aligned}\right\} \tag{2.6}$$

The equation for each ϕ_j has the form

$$\mathrm{L}_0\phi_j = (\text{linear}+\text{nonlinear terms})[\{\phi_k, k<j\}] \equiv R_j \quad (j \geqslant 1, R_1 = 0) \tag{2.7}$$

where L_0 is the self-adjoint operator

$$\mathrm{L}_0 = c\lambda_0^2\omega_0\left(1+\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right). \tag{2.8}$$

With the scalar product in the space of 2π-periodic functions in L^2 (actually we know from the Hopf bifurcation theorem that the above solution is C^2),

$$\langle a,b\rangle = \frac{1}{2\pi}\int_0^{2\pi} a\bar{b}\,\mathrm{d}z \tag{2.9}$$

the solvability conditions for (2.7) read

$$\langle R_j, \mathrm{e}^{\pm \mathrm{i}z}\rangle = 0. \tag{2.10}$$

To render the solution unique, the terms of higher order are required to possess no parts of the solution of the homogeneous equation ($j = 1$), i.e.

$$\langle \phi_j, \mathrm{e}^{\pm \mathrm{i}z}\rangle = 0 \quad \text{for all}\, j > 1. \tag{2.11}$$

For ϕ_1

$$\mathrm{L}_0\phi_1 = 0 \Rightarrow \phi_1 = a\mathrm{e}^{\mathrm{i}z} + \bar{a}\mathrm{e}^{-\mathrm{i}z} \tag{2.12}$$

holds, since we are looking for real solutions here. Imposing the further uniqueness condition

$$\langle \tilde{\phi}, \mathrm{e}^{\mathrm{i}z}\rangle = \epsilon \mathrel{\hat{=}} \langle \phi_1, \mathrm{e}^{\mathrm{i}z}\rangle = 1 \tag{2.13}$$

defines the parameter ϵ and determines the phase factor as $a = 1$. Furthermore we obtain

$$v_1 = -\frac{\omega_0}{1-\phi_0}\phi_1, \quad \lambda_0 p_1' = \alpha\phi_1; \tag{2.14}$$

$$\omega_0^2 = \frac{|G_0|}{F}, \quad \alpha = -\frac{1}{\varphi_0}\left(p_0'\varphi_0' + \frac{\omega_0}{1-\phi_0}\right) = 1 + c\lambda_0^2\omega_0, \tag{2.15}$$

from which ω_0 and λ_0 follow. At the next order,

$$\mathrm{L}_0\phi_2 = \beta\phi_1^2 + \gamma(\phi_1^2)' + \zeta(\phi_1^2)'' \Rightarrow \phi_2 = \varphi_2\mathrm{e}^{2\mathrm{i}z} + \bar{\varphi}_2\mathrm{e}^{-2\mathrm{i}z} + 2\varphi^c, \tag{2.16}$$

$$v_2 = -\frac{\omega_0}{1-\phi_0}\phi_2 - \frac{\omega_0}{(1-\phi_0)^2}\phi_1^2, \quad \lambda_0 p_2' = \alpha\phi_2 - \beta\phi_1^2 \tag{2.17}$$

are obtained, where

$$\left.\begin{aligned} \beta &= \frac{1}{\varphi_0}\left[\frac{p_0'}{2}\varphi_0'' + \frac{\omega_0}{(1-\phi_0)^2} + \alpha\varphi_0'\right], \quad \gamma = \lambda_0\left(\frac{G_0}{1-\phi_0} - \frac{G_0'}{2}\right), \\ \zeta &= -\frac{c\lambda_0^2\omega_0}{1-\phi_0}; \quad \varphi_2 = \frac{4\zeta-\beta-2\mathrm{i}\gamma}{3c\lambda_0^2\omega_0}, \quad \varphi^c = \frac{\beta}{c\lambda_0^2\omega_0}. \end{aligned}\right\} \tag{2.18}$$

Finally, at order ϵ^3 the equation

$$\mathrm{L}_0\phi_3 = a_1\phi_1 + a_2\phi_1' + 2\gamma(\phi_1\phi_2)' + \theta(\phi_1^3)' + 2\beta\phi_1\phi_2 + \eta\phi_1^3 + 2\zeta(\phi_1\phi_2)'' + \frac{\zeta}{1-\phi_0}(\phi_1^3)'' \tag{2.19a}$$

is obtained, with

$$\left.\begin{aligned} a_1 &= \frac{\omega_2}{\varphi_0(1-\phi_0)} + c\lambda_0(2\omega_0\lambda_2 + \lambda_0\omega_2), \quad a_2 = -2F\lambda_0\omega_0\omega_2, \\ \theta &= \lambda_0\left[\frac{G_0}{(1-\phi_0)^2} - \frac{G_0''}{6}\right], \quad \eta = \frac{1}{\varphi_0}\left[\frac{\omega_0}{(1-\phi_0)^3} + p_0'\frac{\varphi_0'''}{6} - \beta\varphi_0' + \alpha\frac{\varphi_0''}{2}\right]. \end{aligned}\right\} \tag{2.19b}$$

According to the solvability condition (2.11), the terms proportional to $\mathrm{e}^{\mathrm{i}z}$ and $\mathrm{e}^{-\mathrm{i}z}$ must vanish, so that

$$\langle \mathrm{L}_0\phi_3, \mathrm{e}^{\mathrm{i}z}\rangle = -\mathrm{i}\lambda_2 + \frac{\mathrm{d}}{\mathrm{d}\omega}\Lambda(\omega_0)\,\omega_2 - \frac{\mathrm{i}a_3}{2c\lambda_0\omega_0} = 0 \tag{2.20a}$$

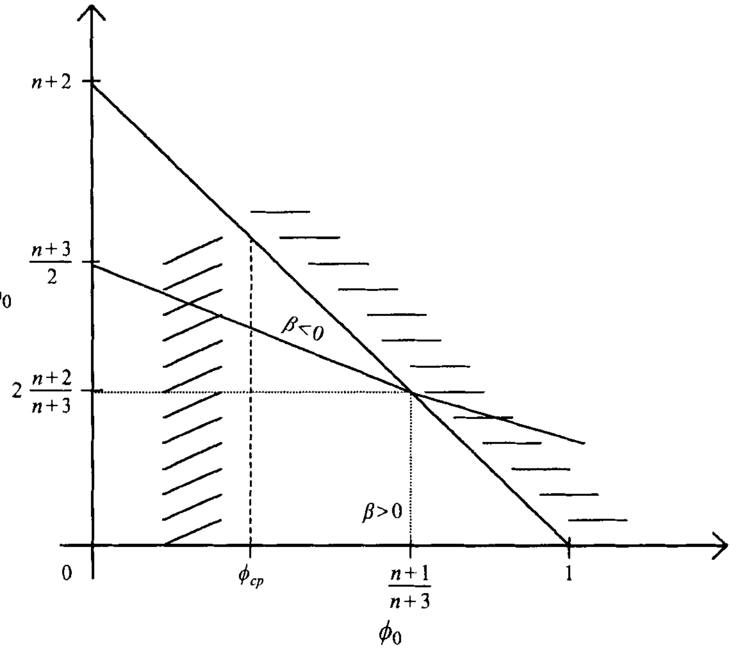

FIGURE 1. The domain in the (ϕ_0, ω_0)-plane, in which a primary bifurcation to a plane vertically travelling wave is possible. The mean voidage increases in the lower subdomain. Voidage values beyond the close-packed limit ϕ_{cp} are not accessible.

and the complex-conjugate equation determine ω_2 and λ_2. Here,

$$a_3 = 2(2\varphi^c + \varphi_2)(\beta - \zeta + i\gamma) + 3i\theta + 3\eta - \frac{3\zeta}{1-\phi_0}, \tag{2.20b}$$

$$\frac{d}{d\omega}\Lambda(\omega)|_{\omega=\omega_0} = -\frac{F}{c} + i\frac{\varphi_0 + p_0'\varphi_0'}{2c\lambda_0\,\omega_0^2\varphi_0} \quad (\Lambda(\omega_0) = i\lambda_0) \tag{2.20c}$$

(the last relation corresponds to expression (5.3) for $\delta = 0$ in Göz 1993*b*).

Before we proceed to the investigation of the transverse stability of this solution, we want to draw some immediate conclusions from the above relations. First of all, since it can be shown (Needham & Merkin 1986) that no periodic solutions of (2.1) exist for $\omega \leqslant 0$, it follows from (2.1*a*) that the particle velocity is always smaller than the propagation speed of the wave. The situation is less clear for the gas velocity, which is given by $\phi(u-\omega) = \phi_0(1-\omega)$. Hence, $u < \omega$ for $\omega > 1$, while $u > \omega$ if $\omega < 1$. We note that an upper limit of ω exists which may well exceed unity (Göz 1993*b*).

Next, we consider the mean values of voidage and particle velocity. From (2.16), (2.18) it is seen that the sign of the mean perturbation from the basic voidage value is given by the sign of the coefficient β, which for the drag force at hand reads

$$\beta = \frac{n+2}{2\phi_0^2}[(n+3)(1-\phi_0) + 2\phi_0 - 2\omega_0].$$

Thus the mean voidage increases, if $\omega_0 < \phi_0 + (n+3)(1-\phi_0)/2$, and decreases otherwise. These conditions are constrained by the result (Göz 1993*b*) that bifurcations of periodic travelling waves may only occur if $(|G_0|/F)^{1/2} = \omega_0 < (n+2)(1-\phi_0)$; for an illustration see figure 1. The latter relation is just a manifestation of the violation of the stability condition on the basic state and roughly means that the drag force must dominate the interparticle forces (cf. Göz 1992).

Furthermore, (2.17) gives

$$\bar{v}_2 = -\frac{2}{c\lambda_0^2(1-\phi_0)}[(1-\phi_0)\beta + c\lambda_0^2\omega_0],$$

which shows that the mean particle velocity is directed downwards, if β is not too negative. On the other hand,

$$\bar{v}_2 > 0 \Leftrightarrow \omega_0 > \frac{(n+2)(1-\phi_0)[(n+3)(1-\phi_0)+4\phi_0]}{2[(n+2)(1-\phi_0)+\phi_0]},$$

which is compatible with the above-mentioned constraint on ω_0 in the physically reasonable range of $\phi_0 < (n+1)/(n+3) \approx 2/3$.

Finally, as it should be, the deviation of the pressure follows exactly that of the voidage, as can be seen from (2.17).

Of course, the discussed behaviour might be altered by larger-amplitude effects.

3. The eigenvalue problem

In this Section we formulate the eigenvalue problem, the solution of which determines the stability of the one-parameter family of periodic one-dimensional waves. The solution is sought in the form of a power expansion of the perturbation variables with respect to the amplitude of the plane wave, which is turn is related to the bifurcation parameter, in accordance with the expansion (2.6). First this will be done for a given wavenumber of the transverse perturbation, which allows us to identify the least-stable modes that are the candidates for driving the secondary instability. Two of these modes are then followed up to higher amplitudes of the primary wave; because this is very technical material, we describe only the results and give the details in the Appendix. The singularity of the perturbed eigenvalue for small transverse wavenumbers indicates that the expansion breaks down in that region and has to be redesigned. This is accomplished by scaling the transverse variables, i.e. transverse velocity and wavenumber, with the square root of the amplitude, from which it turns out that the eigenvalue scales with the amplitude. The resulting new expansion will be carried out in the next section, leading to the ultimate proof of the occurrence of an instability.

3.1. *Set-up*

Linearizing (1.1) at $(\tilde{\phi}, \tilde{v}, p_0' + \tilde{p}')$ (z, ϵ) leads to a system with periodic z-dependent coefficients, so that we may look for a solution of the form

$$(\phi, p, v_y, v_z) = (\psi, q, w, u)(z)\,\mathrm{e}^{\sigma t + \mathrm{i}ky}. \tag{3.1}$$

Here, we assume that $\psi(z, \epsilon)$, etc., are $2\pi/\lambda$-periodic functions, i.e. we allow only for perturbations of the same wavelength. Introducing the scaling of z by λ as in the last section gives the eigenvalue problem

$$-\sigma\psi + \mathrm{i}k(1-\tilde{\phi})\,w + \lambda\,[(1-\tilde{\phi})\,u - (\tilde{v}-\omega)\,\psi]' = 0, \tag{3.2a}$$

$$\mathrm{i}kw + \lambda u' + k^2\varphi(\tilde{\phi})\,q - \lambda\,[\varphi(\tilde{\phi})\,\lambda q' + \varphi'(\tilde{\phi})\,(p_0' + \lambda\tilde{p}')\,\psi]' = 0, \tag{3.2b}$$

$$\begin{aligned} F\sigma(1-\tilde{\phi})\,u - F\omega(1-\phi_0)\,\lambda u' + F\lambda\,[(1-\tilde{\phi})\,u - (\tilde{v}-\omega)\,\psi]\,\tilde{v}' \\ = \psi - \lambda q' - \lambda\,[G(\tilde{\phi})\,\psi]' + \mu(1+\kappa)\,\lambda^2 u'' - \mu k^2 u + \mu\kappa\lambda\,\mathrm{i}kw', \end{aligned} \tag{3.2c}$$

$$\begin{aligned} F\sigma(1-\tilde{\phi})\,w - F\omega\lambda(1-\phi_0)\,w' = -\mathrm{i}kq - \mathrm{i}kG(\tilde{\phi})\,\psi + \mu\kappa\lambda\,\mathrm{i}ku' \\ + \mu\lambda^2 w'' - \mu(1+\kappa)\,k^2 w. \end{aligned} \tag{3.2d}$$

As only ∇p has to be periodic, but not necessarily p itself, q' may assume a non-vanishing mean value in the one-dimensional case, while $\bar{q}$ could be normalized to zero. For $k \neq 0$, however, the equations show that in general $\bar{q} \neq 0$, while $\overline{q'}$ must vanish. Indeed this behaviour can be validated by a general analysis, so that we may write

$$q = q_f + \bar{q} \quad \text{for} \quad k \neq 0, \quad \text{with} \quad \bar{q} = \frac{1}{2\pi}\int_0^{2\pi} q(z)\,\mathrm{d}z \quad \text{and} \quad \bar{q}_f = \overline{q'} = 0. \tag{3.3}$$

Although we shall study (3.2) for arbitrary k, these values will be restricted for finite configurations by

$$k = 2\pi n/(\text{bed width}), \quad n = 0, \pm 1, \pm 2, \ldots. \tag{3.4}$$

We notice that owing to a reflectional symmetry in the horizontal direction the equations are invariant under the transformation $k \to -k$, $w \to -w$. Hence, $w \sim k$, which is also obvious from (3.2d). This means that to each eigenvalue $\sigma(k)$, $k \neq 0$, there belong two eigenfunctions $U_{+k}\mathrm{e}^{\mathrm{i}ky}$ and $U_{-k}\mathrm{e}^{-\mathrm{i}ky}$, with $U_{-k} = U_{+k}(k \to -k, w \to -w)$, such that σ depends on k^2 only and is a semi-simple, double eigenvalue. The total (and real) solution of the linearized equations is then given by the superposition of all such solutions of the type (3.1). In what follows we compute U_{+k}.

According to (2.6) the following expansion is applied:

$$\left.\begin{aligned} \sigma &= \sigma_0 + \epsilon\sigma_1 + \epsilon^2\sigma_2 + \ldots, & \psi &= \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \ldots, \\ q &= q_0 + \epsilon q_1 + \epsilon^2 q_2 + \ldots, & \overline{q'_j} &\sim \delta_{k,0} \quad \text{for all } j, \\ u &= u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots, & w &= w_0 + \epsilon w_1 + \epsilon^2 w_2 + \ldots. \end{aligned}\right\} \tag{3.5}$$

Note that we keep k a fixed, ϵ-independent parameter at this stage. This gives rise to the following sequence of problems:

$$-\sigma_0\psi_j + \mathrm{i}k(1-\phi_0)w_j + \lambda_0(1-\phi_0)u_j' + \lambda_0\omega_0\psi_j' = f_\psi^j, \tag{3.6a}$$

$$\mathrm{i}kw_j + k^2\varphi_0 q_j + \lambda_0 u_j' - \lambda_0^2\varphi_0 q_j'' - \lambda_0 p_0'\varphi_0'\psi_j' = f_q^j, \tag{3.6b}$$

$$[F\sigma_0(1-\phi_0) + \mu k^2]u_j - F\omega_0\lambda_0(1-\phi_0)u_j' - \mu(1+\kappa)\lambda_0^2 u_j'' \\ -\psi_j + \lambda_0 G_0\psi_j' + \lambda_0 q_j' - \mu\kappa\lambda_0\mathrm{i}kw_j' = f_u^j, \tag{3.6c}$$

$$(F\sigma_0 + ck^2)(1-\phi_0)w_j - F\omega_0\lambda_0(1-\phi_0)w_j' - \mu\lambda_0^2 w_j'' - \mu\kappa\lambda_0\mathrm{i}ku_j' \\ + \mathrm{i}kq_j + \mathrm{i}kG_0\psi_j = f_w^j, \tag{3.6d}$$

with

$$f_\psi^0 = f_q^0 = f_u^0 = f_w^0 = 0; \tag{3.7}$$

what is needed from the right-hand sides for $j \in \{1, 2\}$ is given in the Appendix. A single equation for ψ_j can be derived by forming $\lambda_0\partial_z(3.6c) + \mathrm{i}k(3.6d)$ and making use of (3.6a) and (3.6b), namely

$$\mathrm{K}_0\psi_j = -\left(F\sigma_0 + ck^2 + \frac{1}{\varphi_0(1-\phi_0)}\right)f_\psi^j + F\omega_0\lambda_0 f_\psi^{j\prime} + c\lambda_0^2 f_\psi^{j\prime\prime} \\ + f_q^j/\varphi_0 + \lambda_0 f_u^{j\prime} + \mathrm{i}kf_w^j \equiv F^j, \tag{3.8a}$$

where K_0 denotes the operator

$$\mathrm{K}_0 = F\sigma_0^2 + \sigma_0\left(ck^2 + \frac{1}{\varphi_0(1-\phi_0)}\right) + F\omega_0^2 k^2 \\ - \lambda_0[2F\omega_0\sigma_0 + c\omega_0(k^2 - \lambda_0^2)]\frac{\mathrm{d}}{\mathrm{d}z} - \sigma_0 c\lambda_0^2\frac{\mathrm{d}^2}{\mathrm{d}z^2} + c\lambda_0^3\omega_0\frac{\mathrm{d}^3}{\mathrm{d}z^3}. \tag{3.8b}$$

How to then solve (3.6) for the other variables is described in the Appendix.

The unique solvability of (3.8) requires of course appropriate orthogonality conditions on the right-hand side and on ψ_j. These depend on the form of K_0 and thus to some extent on the solution at zeroth order.

3.2. *The zeroth-order terms*

At $O(\epsilon^0)$ the linear stability problem for the uniform state is obtained at the primary bifurcation point (ω_0, λ_0):

$$\mathrm{K}_0 \psi_0 = F^0 = 0. \tag{3.9}$$

Since we admit only periodic perturbations of the same wavelength (and harmonics), we take $\psi_0 \sim \exp(inz)$, $n \in \{0, \pm 1, \pm 2, \ldots\}$, which yields the condition

$$F\sigma_0^2 + \sigma_0\left[c(k^2+n^2\lambda_0^2) + \frac{1}{\varphi_0(1-\phi_0)} - 2inF\lambda_0\omega_0\right] + F\omega_0^2k^2 - inc\lambda_0\omega_0[k^2+(n^2-1)\lambda_0^2] = 0. \tag{3.10}$$

Zero eigenvalues occur for $k = 0$ and $n^2 \in \{0, 1\}$. For $n = 0$ one obtains a constant perturbation of the uniform state, which can be incorporated into the uniform state and thus neglected. (Actually, this is because the basic equations do not determine the base state uniquely, which can be fixed by prescribing appropriate two-dimensional mean values. Demanding, for example, that the two-dimensional mean values of the variables should coincide with those of the one-dimensional periodic solution, forces the two-dimensional mean values of the perturbation to vanish. The latter is automatically fulfilled for $k \neq 0$, but does not represent any problem for pure one-dimensional perturbations either, because in that case $\sigma\bar{\psi} = 0$ follows from (3.2*a*), so that either the one-dimensional mean value of the voidage perturbation vanishes or the perturbation is of neutral stability, regardless of any approximation.)

The value of $n^2 = 1$ corresponds to ϕ_1' as the leading approximation to $\psi = \phi^{0\prime}$, which is a solution of neutral stability and is not important. The next least-stable modes are those with $n^2 \in \{0, 1\}$ and small values of k^2, as can be seen from the expansion

$$\sigma_0 = b_0 + b_1 k^2 + O(k^4) \quad (k^2 \ll 1), \tag{3.11a}$$

$$Fb_0^2 + b_0\left(n^2c\lambda_0^2 + \frac{1}{\varphi_0(1-\phi_0)} - 2inF\lambda_0\omega_0\right) - in(n^2-1)\,c\lambda_0^3\omega_0 = 0, \tag{3.11b}$$

$$b_1 = -\frac{F\omega_0^2 - inc\lambda_0\omega_0 + b_0 c}{2Fb_0 + n^2c\lambda_0^2 + 1/[\varphi_0(1-\phi_0)] - 2inF\lambda_0\omega_0}. \tag{3.11c}$$

It shows that the least-stable (non-vanishing) eigenvalues are obtained for $b_0 = 0$, i.e. $n(n^2-1) = 0$, and small $k^2 \neq 0$, while all other eigenvalues remain an $O(1)$ distance away from 0. This picture is complemented by the behaviour $\sigma_{0,1} = -F\omega_0^2/c + O(k^{-2})$ and $\sigma_{0,2} = -ck^2/F + O(1)$ for large k^2.

For $n = 0$ we obtain $\psi_0 = \text{const.} \neq 0$ as non-trivial solution of (3.9):

$$F\sigma_0^2 + \sigma_0\left(ck^2 + \frac{1}{\varphi_0(1-\phi_0)}\right) + F\omega_0^2k^2 = 0, \quad \psi_0 \equiv \overline{\psi_0}, \tag{3.12a}$$

$$q_0 = \frac{-\sigma_0}{k^2\varphi_0(1-\phi_0)}\psi_0, \quad w_0 = \frac{-i\sigma_0}{k(1-\phi_0)}\psi_0, \quad u_0 = \frac{1}{F\sigma_0(1-\phi_0)+\mu k^2}\psi_0. \tag{3.12b}$$

With regard to the form (3.1) this solution represents a pure transverse mode.

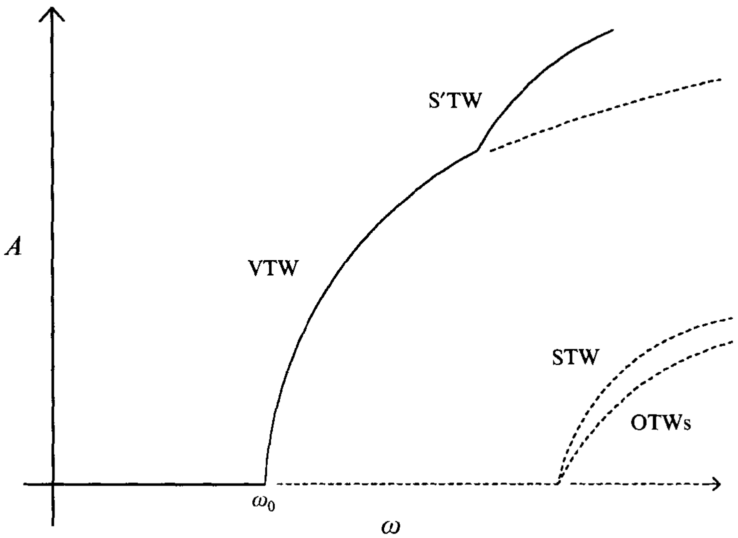

FIGURE 2. Bifurcation scenario with a stationary secondary bifurcation. Solid curves: stable solutions, dashed curves: unstable solutions. VTW: vertically travelling plane wave; OTWs: a pair of oblique travelling waves; S$^{(\prime)}$TW: pairs of horizontally symmetric and antisymmetric vertically travelling waves with transverse structure.

Although k^2 may be arbitrarily small (corresponding to very wide beds), it must not vanish, because otherwise $\sigma_0 \psi_0 = 0$, hence $\psi_0 = 0$ owing to $\sigma_0 u_0 \sim \psi_0$, so that the above solution would cease to exist. Inserting $k = 0$ formally into (3.12*a*), it is seen that the two branches of σ_0 start off for small k^2 near $\sigma_{0,1}(k=0) = 0$ and $\sigma_{0,2}(k=0) = -1/[F\varphi_0(1-\phi_0)]$, respectively. As mentioned above, the one nearest to zero is of greatest interest. Since the first non-vanishing coefficient of the expansion (3.11) is now given by

$$-b_1(n=0) = F\omega_0^2 \varphi_0(1-\phi_0) \equiv \xi_1, \tag{3.13}$$

it has the following series representation for small k^2:

$$\sigma_{0,1} = -\xi_1 k^2 [1 - \varphi_0(1-\phi_0)(c - F\xi_1) k^2 + O(k^4)]. \tag{3.14}$$

For $n^2 = 1$ we obtain 'mixed modes', the expansion of which starts with

$$-b_1(n=\pm 1) = \frac{F\omega_0^2 \mp \mathrm{i}c\lambda_0 \omega_0}{\tilde{c} \mp 2\mathrm{i}F\lambda_0 \omega_0} \equiv \xi_2 \pm \mathrm{i}\xi_3; \quad \tilde{c} \equiv c\lambda_0^2 + \frac{1}{\varphi_0(1-\phi_0)}. \tag{3.15}$$

These modes become unstable when we move along the uniform state by changing ω (and therefore λ), and give rise to a two-dimensional vertically travelling wave and a pair of plane oblique travelling waves as described by Göz (1992), cf. figure 2. Because their point of bifurcation $\omega(k)$ tend towards $\omega_0 = \omega(0)$ for $k \to 0$, it is very likely that they also play a prominent role in the transverse instability of the VTW.

There exists another pure transverse mode, which is connected to the trivial solution $\psi_0 = 0$ of (3.9). It corresponds to pure velocity perturbations of the basic state, as a result of which the velocity field is divergence-free:

$$\psi_0 = 0 \Rightarrow \mathrm{i}kw_0 + \lambda_0 u_0' = 0. \tag{3.16a}$$

In addition,

$$k\overline{w_0} = \sigma_0 \overline{\omega_0} = 0, \quad [F\sigma_0(1-\phi_0) + \mu k^2]\overline{u_0} = -\lambda_0 \overline{q_0'}, \quad q_0' = \text{const.} \times \delta_{k,0}, \tag{3.16b}$$

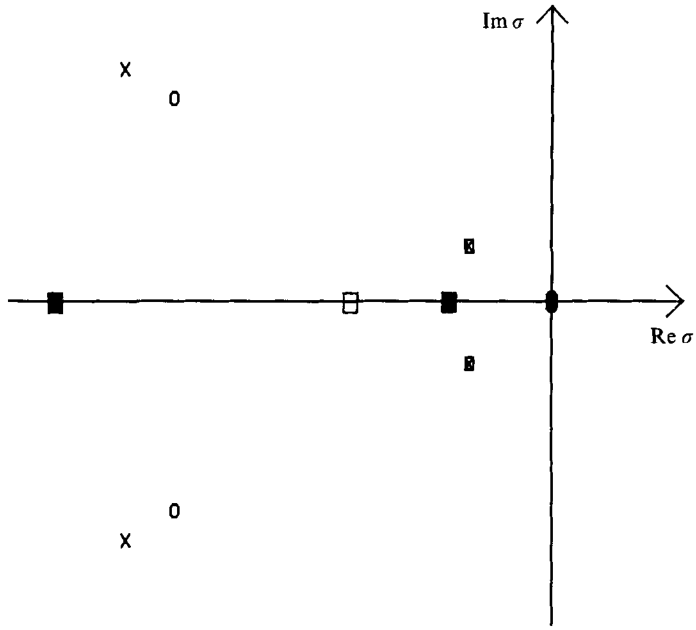

FIGURE 3. An example of the distribution of the least-negative eigenvalues at the primary bifurcation point: ●, one-dimensional mode; ⊗, two-dimensional ('mixed') modes; ■, pure transverse modes; □, pure transverse perturbation of the vertical velocity; ○, one-dimensional second harmonics; ×, two-dimensional second harmonics.

while

$$\left[F\sigma_0(1-\phi_0)+\mu k^2 - F\omega_0\,\lambda_0(1-\phi_0)\frac{\mathrm{d}}{\mathrm{d}z}-\mu\lambda_0^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right]\begin{pmatrix}u_{0,f}\\ w_{0,f}\end{pmatrix}=0, \tag{3.16c}$$

so that for non-vanishing fluctuating parts $u_{0,f}$ or $w_{0,f}$

$$\sigma_0(n) = -\frac{\mu}{F(1-\phi_0)}(k^2+n^2\lambda_0^2)+\mathrm{i}n\lambda_0\,\omega_0. \tag{3.16d}$$

Again, the least-stable non-trivial mode is found for $n=0$ and small k^2:

$$(\psi,q,w)_0=(0,0,0), \quad u_0\equiv\overline{u_0}\neq 0, \quad \sigma_0=-\tilde{\mu}k^2; \quad \tilde{\mu}\equiv\frac{\mu}{F(1-\phi_0)}. \tag{3.17}$$

Obviously, this represents a perturbation consisting of a purely longitudinal velocity depending on the transverse variable only. Modes of type (3.16) have already been noticed by Anderson & Jackson (1968), but have not been considered further because they do not inherit any voidage perturbation at this stage.

To sum up, the least-stable modes are (3.12), (3.17), and the mixed modes belonging to (3.15) – if the transverse wavelength is large enough. The exact meaning of this condition will become clear when we consider the higher-order terms. We anticipate that each one or two of these four modes can become unstable along the VTW; whether it is the pair of mixed modes or one of the pure transverse modes, that is most responsible for the secondary instability, depends on the magnitude of the leading-order coefficients $\tilde{\mu}$, ξ_1, and ξ_2. The distribution of eigenvalues at the primary bifurcation point is sketched in figure 3.

3.3. *The higher-order terms*

Because the transverse modes are simpler than the mixed modes and certainly participate in the generation of a secondary, transverse instability, we follow them up to higher orders in the expansion with respect to the amplitude ϵ of the one-dimensional wave, thereby keeping the transverse wavenumber k of the perturbation a fixed parameter. This involves a good deal of algebra and is therefore given in the Appendix. For both modes the result is that the first-order perturbation of the eigenvalue vanishes, so that the perturbed eigenvalue is of order ϵ^2:

$$\sigma_1 = 0, \quad \sigma = \sigma_0(k^2)+\epsilon^2\sigma_2(\sigma_0, k^2)+O(\epsilon^3). \tag{3.18}$$

The expression for σ_2 is far too complicated to write down here, but details are given in the Appendix (see (A 15) for the mode (3.12) and (A 23) for the mode (3.17)). Now, we should look for a range of parameters in which σ_2 is positive, so that σ has a chance to become zero and eventually positive when the amplitude ϵ becomes large enough. This, however, cannot be accomplished within the expansion (3.18), as it relies on the assumption that the perturbation is much smaller than the leading-order term. On the other hand, we are mainly interested in small k^2 and it turns out that the perturbation of the eigenvalues (3.14) and (3.17) (see (A 16) resp. (A 24)) becomes

$$\sigma_2 = c_1 k^{-2}+O(1) \quad \text{for} \quad k^2 \ll 1. \tag{3.19}$$

Sample numerical computations for the c_1 belonging to (3.14) revealed that its sign alters wildly with changing parameters and if it is positive, σ_2 becomes arbitrarily large when k^2 approaches zero. This singularity can be ascribed to the fact that the transverse modes exist for $k \neq 0$ only. But before that can happen, the expansion (3.18) ceases to be valid, since $\epsilon^2\sigma_2$ exceeds $-\sigma_0$ in the distinguished limit of very long waves and ϵ being kept fixed. Recalling that $\sigma_0 = -c_0 k^2+O(k^4)$ for small wavenumbers, we see that the first two non-vanishing terms in the series (3.18) become of the same order, when k^2 becomes as small as ϵ, and then the eigenvalue is itself of order ϵ:

$$k^2 = c_2\epsilon+O(\epsilon^2) \Rightarrow \sigma = \left(\frac{c_1}{c_2}-c_0 c_2\right)\epsilon+O(\epsilon^2). \tag{3.20}$$

Within this particular scaling it thus becomes possible to reach – and pass – the stability limit. However, the eigenvalues will not be given by a simple relation like (3.20); instead they have to be determined from an equation of fourth degree arising from the interaction between the four least-stable modes identified in the preceding section.

The rest of this section is devoted to ruling out other possible instability mechanisms. From the relation (A 15) for the σ_2 stemming from the mode (3.12) it can be deduced easily that $\sigma_2(\sigma_{0,2}, k^2) = O(1)$ for $k^2 \ll 1$, so that this eigenvalue indeed plays no role in the instability problem. However, one might also worry about the short-wave behaviour of the modes (3.12) and (3.17). In fact, while it can be read from the expressions for σ_2 given in the Appendix that the perturbed eigenvalue belonging to $\sigma_{0,1} = O(1)$ for $k^2 \gg 1$ is also $O(1)$ in this limit (see (A 18)) and thus cannot lead to an instability, the others are positive and of $O(k^4)$, e.g.

$$\sigma_{2,2} = \frac{2c^3}{F(1-\phi_0)^2|-c\lambda_0+\mathrm{i}F\omega_0|^2}k^4+O(k^2), \quad k^2 \gg 1, \tag{3.21}$$

and a very similar expression holds for the σ_2 corresponding to the mode (3.17), see (A 25). Thus, the first two terms in the expansion (3.18) become of the same order, if $k \sim \epsilon^{-1}$, upon which $\sigma \sim \epsilon^{-2}+O(1)$. Employing the scaling $k = \hat{k}/\epsilon$, $\sigma = \hat{\sigma}/\epsilon^2$, and accordingly $w = \hat{w}/\epsilon$ transforms the original system (3.2) into the following set of equations:

$$-\hat{\sigma}\psi+\mathrm{i}\hat{k}\hat{w}(1-\tilde{\phi}) = O(\epsilon^2), \quad \mathrm{i}\hat{k}\hat{w}+\hat{k}^2\varphi(\tilde{\phi})\,q = O(\epsilon^2),$$
$$[F\hat{\sigma}(1-\tilde{\phi})+\mu\hat{k}^2]\,u-\mu\kappa\lambda\,\mathrm{i}\hat{k}\hat{w}' = O(\epsilon^2), \quad [F\hat{\sigma}(1-\tilde{\phi})+\mu(1+\kappa)\hat{k}^2]\,\hat{w} = O(\epsilon^2).$$

Expanding $\hat{k}$, $\hat{\sigma}$, $\hat{w}$ and the other variables as usual with respect to ϵ gives back the relations for large k^2, i.e. either $\hat{\sigma}_0 = -c\hat{k}_0^2/F$ or $\hat{\sigma}_0 = -\mu\hat{k}_0^2/[F(1-\phi_0)]$. Hence, there is no short-wave instability and we have to be concerned only about the long-wave problem, to which we turn now.

4. The rescaled eigenvalue problem

In this section we allow the transverse wavenumber of the perturbation to vary with the amplitude of the plane wave. This enables us to pass the stability limit and reveal the instability mechanism.

4.1. *Rescaling*

According to the above results we implement the scaling

$$k = \epsilon^{1/2}\hat{k}, \quad w = \epsilon^{1/2}\hat{w}, \quad \sigma = \epsilon\hat{\sigma}, \tag{4.1a}$$

and expand these quantities into a power series like the one in (3.5):

$$\hat{k} = k_0+\epsilon k_1+\ldots, \quad \hat{w} = \hat{w}_0+\epsilon\hat{w}_1+\ldots, \quad \hat{\sigma} = \hat{\sigma}_0+\epsilon\hat{\sigma}_1+\ldots. \tag{4.1b}$$

This corresponds to setting $\sigma_0 = 0$, so that the contribution from the eigenvalue disappears from the left-hand side of the equations (3.6). The same happens to the terms proportional to kw_j or k^2, which are transformed into $O(\epsilon)$ terms and thus shifted to the right-hand sides. Except for the first term $\sim w_j$, and the replacement of k with k_0 in the last three terms, the left-hand side of (3.6d) remains unchanged after division by $\epsilon^{1/2}$. Note that the equation for w_j decouples from the others, so at each order one first solves for the other variables and finally determines w_j. How to do that can be read off from the appropriately modified formulae given in the first part of the Appendix. Distinguishing the modified right-hand sides by a hat as well, the following set of equations is obtained:

$$\lambda_0 u_j' = -\frac{\lambda_0\omega_0}{1-\phi_0}\psi_j'+\frac{\hat{f}_\psi^j}{1-\phi_0}, \tag{4.2a}$$

$$\lambda_0^2 q_j'' = \alpha\lambda_0\psi_j'+\frac{\hat{f}_\psi^j}{\varphi_0(1-\phi_0)}-\frac{\hat{f}_q^j}{\varphi_0}, \tag{4.2b}$$

$$-F\omega_0\lambda_0(1-\phi_0)\,u_j'-c\lambda_0^2(1-\phi_0)\,u_j'' = \psi_j-\lambda_0 G_0\psi_j'-\lambda_0 q_j'+\hat{f}_u^j, \tag{4.2c}$$

$$-F\omega_0\lambda_0(1-\phi_0)\,\hat{w}_j'-\mu\lambda_0^2\hat{w}_j'' = -\mathrm{i}k_0(G_0\psi_j+q_j-\mu\kappa\lambda_0 u_j')+\hat{f}_{\hat{w}}^j. \tag{4.2d}$$

The inhomogeneous terms will be given when they are needed. The single equation for ψ_j becomes now

$$\hat{\mathrm{K}}_0\psi_j = -\frac{1}{\varphi_0(1-\phi_0)}\hat{f}_\psi^j+F\omega_0\lambda_0\hat{f}_\psi^{j\prime}+c\lambda_0^2\hat{f}_\psi^{j\prime\prime}+\frac{\hat{f}_q^j}{\varphi_0}+\lambda_0\hat{f}_u^{j\prime} \equiv \hat{F}^j, \tag{4.3a}$$

with the reduced operator

$$\hat{\mathrm{K}}_0 = \lambda_0 \mathrm{L}_0 \frac{\mathrm{d}}{\mathrm{d}z} = c\lambda_0^3 \omega_0 \left(1+\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right)\frac{\mathrm{d}}{\mathrm{d}z}. \tag{4.3b}$$

Averaging the equations (4.2) leads to

$$\overline{\hat{f}^j_\psi} = 0, \quad \overline{\hat{f}^j_q} = 0, \quad \overline{\psi_j}+\overline{\hat{f}^j_u} = 0, \quad \mathrm{i}k_0(G_0\overline{\psi_j}+\overline{q_j}) = \overline{\hat{f}^j_{\hat{w}}}. \tag{4.4}$$

The solvability conditions for (4.3) are given by $\langle \hat{F}^j, \mathrm{e}^{\pm\mathrm{i}z}\rangle = 0$ for all $j > 0$ and $\overline{\hat{F}^j} = 0$, the latter of which is implied by (4.4). We assume $k_0 \neq 0$, which will give us eigenvalues $\mathrm{Re}\,\hat{\sigma}_0 > 0$, thus proving the existence of an instability.

4.2. *The new expansion*

To leading approximation we get $\mathrm{L}_0\psi_0' = \hat{F}^0 = 0$ and $\overline{\psi_0} = \overline{q_0} = 0$, so that

$$\psi_0 = \psi_0^+ \mathrm{e}^{\mathrm{i}z} + \psi_0^- \mathrm{e}^{-\mathrm{i}z}. \tag{4.5a}$$

Furthermore,

$$u_0 = \overline{u_0} - \frac{\omega_0}{1-\phi_0}\psi_0, \quad \lambda_0 q_0' = \alpha\psi_0 \tag{4.5b}$$

is obtained from (4.2*a, b*), hence (4.2*d*) gives

$$\hat{w}_0 = \overline{\hat{w}_0} + \mathrm{i}k_0(\hat{w}_0^+\mathrm{e}^{\mathrm{i}z}+\hat{w}_0^-\mathrm{e}^{-\mathrm{i}z}), \quad \hat{w}_0^+ = (f+\mathrm{i}g)\psi_0^+, \quad \hat{w}_0^- = (f-\mathrm{i}g)\psi_0^-, \tag{4.6a}$$

with

$$f+\mathrm{i}g = \frac{-F\omega_0(1-\phi_0)+\mathrm{i}\lambda_0[\mu+\mu^2\lambda_0^2\omega_0/(1-\phi_0)+F^2\omega_0^3(1-\phi_0)]}{(\mu\lambda_0^2)^2+[F\omega_0\lambda_0(1-\phi_0)]^2}. \tag{4.6b}$$

At the next order,

$$\hat{f}^1_\psi = \hat{\sigma}_0\psi_0 - \mathrm{i}k_0(1-\phi_0)\hat{w}_0 + \lambda_0(\phi_1 u_0 + v_1\psi_0)', \tag{4.7a}$$

$$\hat{f}^1_q = -k_0^2\varphi_0 q_0 - \mathrm{i}k_0\hat{w}_0 + \lambda_0^2\varphi_0'(p_1'\psi_0+\phi_1 q_0')' + \lambda_0 p_0'\varphi_0''(\phi_1\psi_0)', \tag{4.7b}$$

$$\hat{f}^1_u = -[F\hat{\sigma}_0(1-\phi_0)+\mu k_0^2]u_0 + \mathrm{i}k_0\mu\kappa\lambda_0\hat{w}_0' - F\lambda_0(1-\phi_0)\overline{u_0}v_1' - \lambda_0 G_0'(\phi_1\psi_0)', \tag{4.7c}$$

$$\hat{f}^1_{\hat{w}} = -(1-\phi_0)(F\hat{\sigma}_0+ck_0^2)\hat{w}_0 - \mathrm{i}k_0 G_0'\phi_1\psi_0 - \mathrm{i}k_1(q_0+G_0\psi_0-\mu\kappa\lambda_0 u_0'). \tag{4.7d}$$

The averaged relations (4.4) yield

$$\overline{\psi_1} = [F\hat{\sigma}_0(1-\phi_0)+\mu k_0^2]\overline{u_0}, \tag{4.8a}$$

$$\overline{\hat{w}_0} = 0, \quad \overline{q_1}+G_0\overline{\psi_1}+G_0'\overline{\phi_1\psi_0} = 0, \tag{4.8b}$$

while the other solvability conditions give

$$[\hat{\sigma}_0+(\xi_2+\mathrm{i}\xi_3)k_0^2]\psi_0^+ + \mathrm{i}\lambda_0\overline{u_0} = 0, \tag{4.9a}$$

$$[\hat{\sigma}_0+(\xi_2-\mathrm{i}\xi_3)k_0^2]\psi_0^- - \mathrm{i}\lambda_0\overline{u_0} = 0. \tag{4.9b}$$

Here we refind the coefficients $\xi_{2,3}$ characterizing the mixed modes of §3, see (3.15).

At $O(\epsilon^2)$ we need only the first two relations of the averaged equations (4.4). Taking account of $\overline{\psi_0} = \overline{q_0} = \overline{\hat{w}_0} = 0$ gives us the following relationships between the $O(\epsilon)$ mean values:

$$\hat{\sigma}_0\overline{\psi_1} - \mathrm{i}k_0(1-\phi_0)\overline{\hat{w}_1} + \mathrm{i}k_0\overline{\phi_1\hat{w}_0} = 0, \tag{4.10a}$$

$$k_0^2\varphi_0\overline{q_1} + \mathrm{i}k_0\overline{\hat{w}_1} + k_0^2\varphi_0'\overline{\phi_1 q_0} = 0. \tag{4.10b}$$

We first eliminate $\overline{\hat{w}_1}$ from these two equations and then use (4.8*b*) to eliminate $\overline{q_1}$; this leaves us with an expression for $\overline{\psi_1}$:

$$(\hat{\sigma}_0+\xi_1 k_0^2)\overline{\psi_1} + \mathrm{i}k_0\overline{\phi_1\hat{w}_0} + k_0^2(1-\phi_0)(\varphi_0'\overline{\phi_1 q_0} - \varphi_0 G_0'\overline{\phi_1\psi_0}) = 0, \tag{4.11}$$

in which we find the main coefficient ξ_1 of one of the transverse modes, see (3.13), (3.14). Inserting (4.8a) into (4.11) gives another relation between $\overline{u_0}$, ψ_0^+, and ψ_0^- in addition to (4.9a,b), namely

$$(\hat{\sigma}_0+\xi_1 k_0^2)[\hat{\sigma}_0 F(1-\phi_0)+\mu k_0^2]\overline{u_0} = k_0^2[(\xi_4+\mathrm{i}\xi_5)\psi_0^+ +(\xi_4-\mathrm{i}\xi_5)\psi_0^-], \qquad (4.12)$$

with

$$\xi_4 = f+G_0'\varphi_0(1-\phi_0), \quad \xi_5 = g+\alpha\varphi_0'(1-\phi_0)/\lambda_0. \qquad (4.13)$$

These coefficients are obviously connected to the transverse velocity perturbation and appear also in the long-wave expressions for the perturbed eigenvalues of the two transverse modes, see (A 16), (A 24a).

4.3. *The instability mechanism*

The three equations (4.9a), (4.9b) and (4.12) for the three unknowns $\overline{u_0}$, ψ_0^+, and ψ_0^- possess non-trivial solutions, if and only if the following relations between $\hat{\sigma}_0$ and k_0^2 holds:

$$(\hat{\sigma}_0+\xi_1 k_0^2)(\hat{\sigma}_0+\tilde{\mu}k_0^2)[(\hat{\sigma}_0+\xi_2 k_0^2)^2+\xi_3^2 k_0^4]-\frac{2\lambda_0}{F(1-\phi_0)}k_0^2[\xi_5\hat{\sigma}_0+k_0^2(\xi_2\xi_5-\xi_3\xi_4)] = 0. \qquad (4.14)$$

It is important to note that a non-vanishing $\overline{u_0}$ is decisive for getting a non-trivial solution, since otherwise $\psi_0 = 0$ etc. or $\hat{\sigma}_0 = k_0 = 0$. This means that the modes participating in the generation of the secondary instability consist of $\{\cos z\cos(ky), \sin z\cos(ky), \cos z\sin(ky), \sin z\sin(ky)\}$ in all perturbation variables plus pure transverse modes $\{\cos(ky), \sin(ky)\}$ in the vertical particle velocity. In addition, a short calculation based on the linearization of (1.1d) shows that the gas velocity behaves similarly:

$$(u_y^{\mathrm{gas}}, u_z^{\mathrm{gas}}) = (\epsilon^{1/2}\hat{\eta},\rho)\,\mathrm{e}^{\epsilon\hat{\sigma}t+\mathrm{i}\epsilon^{1/2}\hat{k}y} \Rightarrow \hat{\eta}_0 = \hat{w}_0+\frac{\mathrm{i}k_0 q_0}{p_0'}, \quad \rho_0 = \overline{u_0}+\frac{\omega_0-1}{\phi_0}\psi_0. \qquad (4.15)$$

Thus, averaging the eigenvector with respect to the vertical coordinate leaves only a vertical component of the velocity, $\overline{u_0}\mathrm{e}^{\mathrm{i}ky}$, which points alternatingly up- and downwards when changing horizontal position. The instability evaluated below is therefore of 'overturning' type in the sense of Batchelor & Nitsche (1991), see also Batchelor (1933). However, the details appear to be rather involved, as can be seen from the expressions (4.5) and (4.6) for the eigenvectors, and from (4.9) and (4.12) for the coefficients entering them. In general, there will be a phase shift between the primary wave and the perturbation. To see this, let us consider a real eigenvalue $\hat{\sigma}_0$. Then $\overline{u_0}$ is also real, while $\hat{w}_0 = \mathrm{i}k_0\tilde{w}$, where $\tilde{w}$ is a real function. To get a real ψ_0, and thus a real u_0, one has to choose $\psi_0^+ = \exp(-\mathrm{i}\theta)$, where θ is the argument of the complex number $\hat{\sigma}_0+(\xi_2+\mathrm{i}\xi_3)k_0^2$. Then $\psi_0 = 2\cos(z-\theta)$ is obtained, while $\phi_1 = 2\cos z$. In addition, the transverse velocity inherits another phase shift: $\tilde{w} \sim \cos(z-\theta')$, with $\theta' \neq \theta$.

Things become even more complicated if the eigenvalue is not real but complex. As our main interest lies in the analytical proof of the occurrence of a secondary instability, and because a numerical bifurcation analysis has been completed recently, we shall not go into more detail here, but instead refer to the work of Glasser, Kevrekidis & Sundaresan (1995). Nevertheless, at the end of this paper we shall describe the form of the bifurcating solutions using only symmetry arguments and the information about the critical eigenvalue(s).

Proceeding now with the investigation of (4.14), it is almost immediately seen that the limiting behaviour for $k_0^2 \to \infty$ of the four eigenvalues $\hat{\sigma}_0$ coincides with that for $k^2 \to 0$ of the eigenvalues σ_0 of the four least-stable modes identified in section §3.2:

$$\hat{\sigma}_{0,1} \sim -\xi_1 k_0^2, \quad \hat{\sigma}_{0,2} \sim -\tilde{\mu} k_0^2; \tag{4.16a}$$

$$\hat{\sigma}_{0;3,4} \sim -(\xi_2 \pm \mathrm{i}\xi_3) k_0^2; \quad k_0^2 \gg 1, \tag{4.16b}$$

cf. (3.12), (3.17) and (3.15). This has to be so owing to the scaling (4.1), and gives us stability for large k_0^2, since $\xi_1 > 0$ is obvious from (3.14), while $\xi_2 > 0$ can be checked from (3.15). Next, we consider the limit of small k_0^2. For that purpose it is convenient to rewrite (4.14) in the form

$$P(\hat{\sigma}_0) \equiv \hat{\sigma}_0^4 + \chi_1 \hat{\sigma}_0^3 k_0^2 + \chi_2 \hat{\sigma}_0^2 k_0^4 + \chi_3 \hat{\sigma}_0 k_0^6 + \chi_4 k_0^8 - \chi_5 \hat{\sigma}_0 k_0^2 - \chi_6 k_0^4 = 0, \tag{4.17a}$$

with

$$\left.\begin{aligned} &\chi_1 = \tilde{\mu} + \xi_1 + 2\xi_2, \quad \chi_2 = \tilde{\mu}(\xi_1 + 2\xi_2) + \xi_2^2 + \xi_3^2 + 2\xi_1\xi_2, \\ &\chi_3 = \tilde{\mu}(\xi_2^2 + \xi_3^2 + 2\xi_1\xi_2) + \xi_1(\xi_2^2 + \xi_3^2), \quad \chi_4 = \tilde{\mu}\xi_1(\xi_2^2 + \xi_3^2), \\ &\chi_5 = \frac{2\lambda_0 \xi_5}{F(1-\phi_0)}, \quad \chi_6 = \frac{2\lambda_0}{F(1-\phi_0)}(\xi_2\xi_5 - \xi_3\xi_4). \end{aligned}\right\} \tag{4.17b}$$

Obviously, all branches of eigenvalues emanate from the origin; in the vicinity they expand like

$$\hat{\sigma}_0 = A k_0^{2/3} + B k_0^2 + C k_0^{10/3} + O(k_0^{14/3}) \quad (k_0 < 1), \tag{4.18}$$

where the coefficients are determined by

$$\begin{aligned} &A(A^3 - \chi_5) = 0, \quad B(4A^3 - \chi_5) = \chi_6 - \chi_1 A^3, \\ &C(4A^3 - \chi_5) = -A^2(\chi_2 + 3\chi_1 B + 6B^2). \end{aligned} \tag{4.19}$$

There are four solutions for A:

$$A_1 = (\chi_5)^{1/3}, \quad A_2 = 0, \quad A_{3,4} = (-1 \pm \mathrm{i}\sqrt{3}) A_1/2, \tag{4.20}$$

and these determine the expansion completely. For the further evaluation we note that $\xi_5 > 0$, since $g > 0$ (cf. (4.6b)) and $\varphi_0' > 0$ (cf. (2.15); p_0' is negative owing to (2.2)). Hence $\chi_5 > 0$, and the same is obviously true for the other coefficients (4.17b) except possibly the last one, χ_6. But this means $A_1 > 0$, so that this branch becomes positive, while the real part of the third and fourth eigenvalues becomes strongly negative near the origin. The real part of the latter pair is therefore unlikely to become positive for any value of k_0 and we shall give more evidence for this below. The second branch differs from the others in that certain terms in the expansion (4.18) vanish; it can be proved that it behaves as

$$\hat{\sigma}_0^{(2)} = -(\chi_6/\chi_5) k_0^2 + O(k_0^6) \quad (k_0^2 < 1), \quad \chi_6/\chi_5 = (\xi_2\xi_5 - \xi_3\xi_4)/\xi_5. \tag{4.21}$$

Now, except at the origin, $\hat{\sigma}_0$ can vanish only at

$$k_{0,0}^4 = \chi_6/\chi_4 \sim \xi_2\xi_5 - \xi_3\xi_4, \tag{4.22}$$

provided that $\chi_6 > 0$. In turn, at this $k_{0,0}$ at most one eigenvalue can vanish, because the product of the remaining eigenvalues must equal $(\chi_5 - \chi_3 k_{0,0}^4) k_{0,0}^2 = (\chi_4\chi_5 - \chi_3\chi_6) k_{0,0}^2/\chi_4$, which is in general not zero. In fact, the sign of this number determines how an eigenvalue crosses zero: the slope at $k_{0,0}$ is given by

$$\hat{\sigma}_{0,0}' = 4k_{0,0} \frac{\chi_4\chi_6}{\chi_4\chi_5 - \chi_3\chi_6} \quad (\text{if} \quad \chi_6 > 0). \tag{4.23}$$

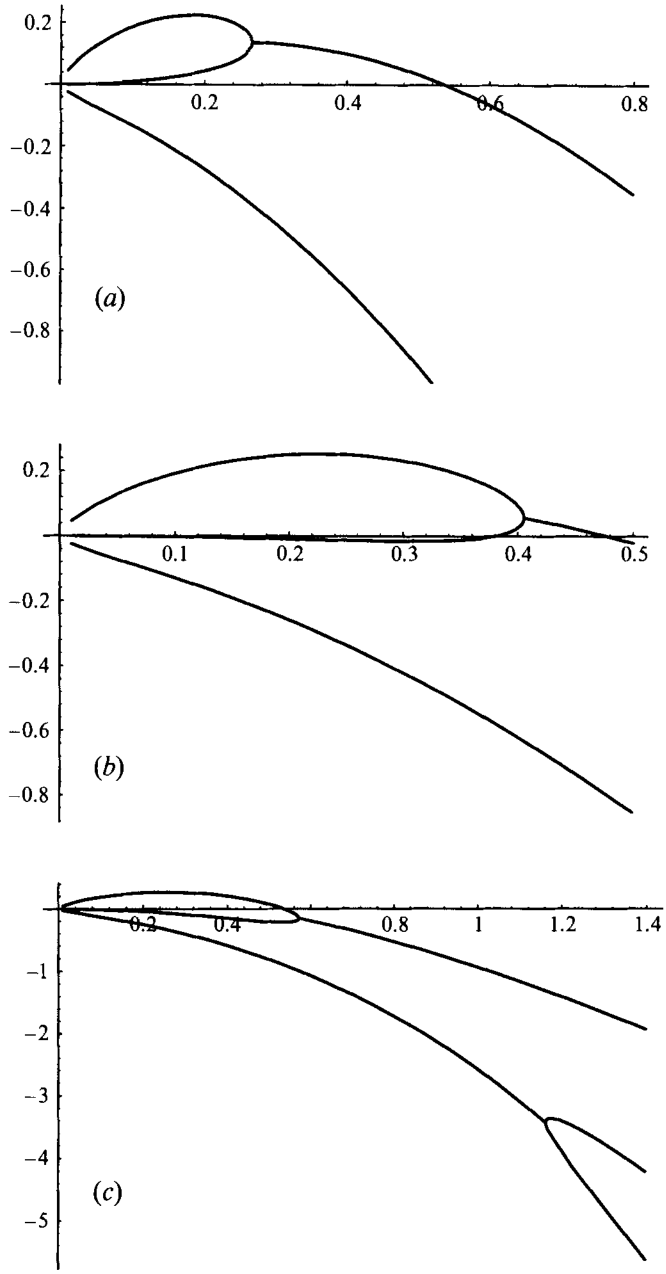

FIGURE 4. Typical behaviour of the eigenvalues for the case $\tilde{\mu}+\xi_1 > 2\xi_2$. Only the real part is shown; a merging of two branches indicates that these eigenvalues become a complex-conjugate pair. Sample parameters are $\xi_1 = 2$, $\xi_2 = 1$, $\tilde{\mu} = 3$, $\xi_3 = 1$, $\xi_5 = F(1-\phi_0)/(2\lambda_0)$, hence $\chi_1 = 7$, $\chi_2 = 18$, $\chi_3 = 22$, $\chi_4 = 12$, $\chi_5 = 1$, and (*a*) $\chi_6 = -1$, (*b*) $\chi_6 = 0.25$, (*c*) $\chi_6 = 1$.

Hence, if χ_6 is negative, two real positive eigenvalues exist near the origin, a $k^{2/3}$-branch and the k^2-branch. Because neither one can become zero for any $k_0 \neq 0$, but tend to $-\infty$ (either on the real axis or in the complex plane), these two branches must merge, turn into a pair of complex-conjugate eigenvalues, and as such cross the axis $\operatorname{Re}\hat{\sigma}_0 = 0$ (cf. figure 4*a*). For the special case $\chi_6 = 0$ the scenario is effectively the same, except that the k^2-branch becomes a k^6-branch: $\hat{\sigma}_0^{(2)} = (\chi_4/\chi_5)\,k^6 + o(k^6)$. If χ_6 is positive but small, the k^2-branch starts off negative from zero, but then turns and crosses the axis $\hat{\sigma}_0 = 0$, before it merges again with the $k^{2/3}$-branch. This is due to $\hat{\sigma}'_{0,0} > 0$ for $0 < \chi_6 < \chi_4 \chi_5 / \chi_3$ (figure 4*b*). The merging point wanders towards the point $(\hat{\sigma}_0, k_0) = (0, k_{0,0})$ and $\chi_4 \chi_5 - \chi_3 \chi_6 \rightarrow 0^+$, which is indicated by the behaviour $\hat{\sigma}'_{0,0} \rightarrow \infty$. For larger values of χ_6 the merging of the two branches takes place below the axis, i.e. now it is the primarily real and positive $k^{2/3}$-branch that crosses the line $\hat{\sigma}_0 = 0$ with a negative slope (figure 4*c*). In the last case, there is only one positive (and real) eigenvalue for all $k_0 \in (0, k_{0,0})$.

If the two eigenvalues under consideration stay complex conjugate for $k_0 \rightarrow \infty$, they must behave as in (4.16*b*); otherwise they break up again into two real eigenvalues with the limiting behaviour (4.16*a*).

Before we come to the linkage of the asymptotic behaviour of the various branches, we wish to point out that the scenarios described can also be validated by examining the polynomial $P(\hat{\sigma}_0)$ defined in (4.17*a*). One has simply to observe that

$$P(0) > 0 \Leftrightarrow k_0^4 > \chi_6/\chi_4, \quad P'(0) > 0 \Leftrightarrow k_0^4 > \chi_5/\chi_3, \quad P''(0) > 0 \quad \text{for} \quad \hat{\sigma}_0 \geqslant 0, \tag{4.24}$$

and vary χ_6 as above, taking k_0 as a parameter. Although this consideration yields no quantitative information about the behaviour of the various branches for small and large k_0, it shows clearly that no eigenvalues occur with positive real part other than those already found.

Finally, in order to identify the most unstable mode(s), it is necessary to link the behaviour of the different branches for small k_0 to that for large k_0. This has been achieved with the help of the symbolic software package *Mathematica* and lead to the results summarized in table 1. It is seen that if the average of the coefficients ξ_1 and $\tilde{\mu}$, characterizing the pair of initially pure transverse modes (3.12) and (3.17), is larger than the real part of the coefficient ξ_2 of the pair of mixed modes (3.15), then the latter is the dominating one and thus responsible for the secondary instability. This is conceivable, because in this case the mixed modes are less stable than the combination of the transverse modes (in the sense of the leading-order approximation in the transverse wavenumber at the onset of the VTW). The possible instability scenarios for this case are shown in figure 4(*a*–*c*).

In the opposite case, where $\xi_1 + \tilde{\mu} < (\xi_2 + \mathrm{i}\xi_3) + (\xi_2 - \mathrm{i}\xi_3) = 2\xi_2$, the pair of transverse modes dominates over the pair of mixed modes, and the initially least stable of the transverse modes develops into the most unstable perturbation of the one-dimensional wave. In this case the upper (double) branch of complex-conjugate eigenvalues appearing in figure 4 above a certain value of k_0 breaks up again into two branches of real eigenvalues at a larger k_0, while the lower branch either represents a pair of complex-conjugate eigenvalues with negative real part for all transverse wavenumbers as suggested in figure 4(*a*, *b*) or breaks up into two real eigenvalues as in figure 4(*c*) but only for an intermediate range of k_0.

That both cases are possible in principle, can be shown by considering the limits $\omega_0 \rightarrow 0^+$ and $\omega_0 \rightarrow d^-$. We know (Göz 1993*b*, see also §2) that primary bifurcations occur only for positive wave speeds smaller than $d = -\phi_0(1 + p'_0 \varphi'_0/\varphi_0) \doteq (n+2)(1-\phi_0)$. Since wavelength and wave speed at the onset of the VTW are related by (2.15), we can eliminate λ_0^2 to get

$$\tilde{\mu} + \xi_1 - 2\xi_2 = \tilde{\mu} + F\omega_0^2 \phi_0 - \frac{2Fc\phi_0 \omega_0^3 (3d - 2\omega_0)}{cd^2 + 4F^2 \phi_0 \omega_0^3 (d - \omega_0)}. \tag{4.25}$$

For small ω_0 this approaches $\tilde{\mu} > 0$, while for $\omega_0 \rightarrow d$ it is reduced to

$$\tilde{\mu} - F\phi_0 d^2 \doteq F[(FR)^{-1} - (n+2)^2 \phi_0 (1-\phi_0)^3].$$

$\lim_{k_0 \to 0} \hat{\sigma}_0/k_0^{2/3}$	$\lim_{k_0 \to \infty} \hat{\sigma}_0/k_0^2$	
	$\tilde{\mu}+\xi_1 > 2\xi_2$	$\tilde{\mu}+\xi_1 < 2\xi_2$
$A_1 > 0$	$-\xi_2 \pm \mathrm{i}\xi_3$	$-\min(\tilde{\mu}, \xi_1)$
$A_2 = 0$	$-\xi_2 \mp \mathrm{i}\xi_3$	$-\max(\tilde{\mu}, \xi_1)$
$A_3 \in \mathbb{C}$	$-\min(\tilde{\mu}, \xi_1)$	$-\xi_2 \pm \mathrm{i}\xi_3$
$A_4 = \bar{A}_3$	$-\max(\tilde{\mu}, \xi_1)$	$-\xi_2 \mp \mathrm{i}\xi_3$

TABLE 1. The asymptotic behaviour of the four eigenvalues $\hat{\sigma}_0$ for small and large k_0^2

The last expression, obtained using the drag law (1.2), becomes negative for a large enough product *FR*, which can be achieved, for instance, with a small enough particle-phase viscosity or a large enough particle diameter. In fact, tracing back the origin of the secondary instability to either the mixed mode or one of the transverse modes (namely (3.12)), Glasser *et al.* (1995) could validate both scenarios.

In any case, the influence of the transverse modes is evident from the important contribution of a non-vanishing vertical mean value of the vertical velocity, $\overline{u_0}$, to the leading order of the rescaled expansion which otherwise shows mixed-mode behaviour in all variables. Such a contribution is present neither in the mixed modes (3.15) nor in their $O(\epsilon)$-perturbation, as can be easily deduced from averaging the equations (A 1*a–d*). Instead it is a characteristic feature of the transverse modes (3.12) and, particularly, (3.17).

5. Conclusions

We have followed a one-dimensional periodic vertically travelling wave (VTW) solution up to a small amplitude ϵ, and have calculated its linear stability in the vicinity of the primary bifurcation point to two-dimensional disturbances of the same longitudinal, but large ($\sim 1/\epsilon^{1/2}$) transverse wavelength. The occurrence of a secondary instability could be attributed to the interaction of the plane wave with a disturbance packet consisting of a pair of mixed (i.e. two-dimensional) modes and two initially pure transverse modes. Two scenarios are possible: either the interaction of the plane wave with the pair of transverse modes is more efficient in feeding energy into the mixed modes and drives one or both of them across the border of stability, or vice versa. Which one is realized depends on whether the (negative) sum of the (real) eigenvalues of the transverse modes is smaller or larger than the (negative) sum of the (complex-conjugate) eigenvalues of the mixed modes, i.e. which pair of modes is less stable at the onset of the VTW (cf. (3.13), (3.14), (3.15), (3.17) and table 1). This should be contrasted with our previous finding (Göz 1992) that the mixed modes become unstable along the uniform base state as well (where the transverse wavenumber need not be small), while the pure transverse perturbations of the uniform state remain always stable.

First, the development of two of the four least-stable perturbation modes has been evaluated along the VTW branch for arbitrary finite transverse wavenumbers (namely that of the two initially pure transverse modes for relative simplicity). The calculations presented in the Appendix show clearly, how the transverse modes develop into mixed modes with an additional vertical velocity component, by interaction with the first and second harmonics of the plane VTW. This expansion is not capable of yielding an instability, but its breakdown for small (and, less important, large) transverse wavenumbers indicates how the transverse variables have to be scaled with respect to the amplitude of the VTW. The redesigned expansion include all four perturbation modes in like manner, and their interaction with the first harmonic of the VTW produces either one or two eigenvalues with positive real part within a certain range of transverse wavenumbers, $k \in (0, \epsilon^{1/2} k_c)$ say. The scaling of the critical k^2 with the amplitude means that the narrower the fluidized bed the larger the amplitude to which the plane wave can grow without becoming unstable to transverse perturbations, until eventually the approximation of the VTW based on the expansion of §2 ceases to be valid.

To state it in another manner, consider a VTW of given amplitude ϵ_0. It will be (transversally) stable for beds narrower than $2\pi/(\epsilon_0^{1/2} k_c)$, but lose its stability to the described disturbance packet, if the bed can accommodate waves with transverse wavenumber $\epsilon_0^{1/2} k_c$. Note that because it is proportional to k, the horizontal velocity scales with the square root of the amplitude, too. The nature of this instability depends on the behaviour of the corresponding critical eigenvalues. If they cross the imaginary axis as a complex-conjugate pair, the instability sets in as an oscillatory one; if they split into two real eigenvalues before crossing, one becoming positive and the other staying negative, then a stationary instability will occur.

In principle, the one-dimensional wavetrain $\phi(z)$ may undergo a transition either to another one-dimensional but now time-dependent solution $\phi(z, t)$, which would then be quasi-periodic in the laboratory frame, or to a two-dimensional pattern. Here we have concentrated our investigations on the second case; the one-dimensional stability has been studied by Needham & Merkin (1986) by other means. Performing the analysis of §3 for the one-dimensional case ($k = 0$) would yield an eigenvalue perturbation of order ϵ^2, the sign of which determines the one-dimensional stability of the VTW (because it is a perturbation of the zero eigenvalue). According to the results of Needham & Merkin, this will depend on the wavelength of the VTW. However, since the pure one-dimensional eigenvalue is of higher order than the others considered here, which scale with the amplitude ϵ, the transverse instability should be the dominating one (for small amplitudes, that is to say, close to the primary bifurcation point, as always in this paper).

The wave patterns emanating owing to the two types of transverse instability can be derived from symmetry arguments (Golubitsky, Stewart & Schaeffer 1985). The equations as well as the basic state possess translational symmetries in time and in both spatial variables, which are recast into rotational symmetries by looking for solutions periodic in time and space (in a vertically moving coordinate system); in the horizontal direction an additional reflectional symmetry is present (cf. Göz 1992). Now the VTW breaks the rotational symmetry in z only and leaves the horizontal symmetries intact. Therefore, the eigenvalues σ of the linearization at a VTW depend on k^2 only. For a stationary bifurcation from the VTW, i.e. when the largest eigenvalue is real and crosses zero at criticality, $\sigma_c(k_c) = 0$, the corresponding eigenvectors consist of the set $\{U_{k_c}(z)\exp(\mathrm{i}k_c y), U_{-k_c}(z)\exp(-\mathrm{i}k_c y)\}$, where the $U_{\pm k_c}$ are periodic in $z = x - \omega t$. The normalization can be chosen such that U_{k_c} has real components except for the horizontal velocity component which is purely imaginary, so that $U_{-k_c} = \bar{U}_{k_c}$ (see §4.3). This gives the symmetric and antisymmetric combinations

$$U_{k_c}(z)\,\mathrm{e}^{\mathrm{i}k_c y} + U_{-k_c}(z)\,\mathrm{e}^{-\mathrm{i}k_c y} \sim ((\psi_0, q_0', u_0)\cos(k_c y)), \epsilon^{1/2} k_c \tilde{w}_0 \sin(k_c y)),$$

$$-\mathrm{i}(U_{k_c}(z)\,\mathrm{e}^{\mathrm{i}k_c y} - U_{-k_c}(z)\,\mathrm{e}^{-\mathrm{i}k_c y}) \sim ((\psi_0, q_0', u_0)\sin(k_c y)), \epsilon^{1/2} k_c \tilde{w}_0 \cos(k_c y))$$

at the onset. Complemented by higher-order terms, this yields two branches of quasi-stationary two-dimensional waves $U(z,y)$, which differ only in a phase shift by half a period in the horizontal direction.

For a Hopf bifurcation $\sigma_c = \pm \mathrm{i}s_0, s_0 = \epsilon|\operatorname{Im}\hat{\sigma}_0(k_c^2)|$ $(\operatorname{Re}\hat{\sigma}_0(k_c) = 0)$, with the critical modes $U_{k_c}(z)\exp(\mathrm{i}k_c y + \mathrm{i}s_0 t)$, $U_{-k_c}(z)\exp(-\mathrm{i}k_c y + \mathrm{i}s_0 t)$ and their complex conjugates; now the components of $U_{\pm k_c}$ are complex periodic functions of z. Again symmetric and antisymmetric combinations can be formed:

$$\mathrm{e}^{\mathrm{i}s_0 t}(U_{k_c}\mathrm{e}^{\mathrm{i}k_c y} \pm U_{-k_c}\mathrm{e}^{-\mathrm{i}k_c y}), \quad \mathrm{e}^{-\mathrm{i}s_0 t}(\bar{U}_{k_c}\mathrm{e}^{-\mathrm{i}k_c y} \pm \bar{U}_{-k_c}\mathrm{e}^{\mathrm{i}k_c y}).$$

The real combinations of the terms containing either the plus or minus signs lead to symmetric and antisymmetric 'standing travelling waves' $U(z,y,t)$, i.e. vertically travelling waves with horizontally oscillating amplitudes. In addition, we obtain what we may call rotating travelling waves of the form

$$U_{k_c}\mathrm{e}^{\mathrm{i}(s_0 t + k_c y)} + \text{c.c.}, \quad U_{-k_c}\mathrm{e}^{\mathrm{i}(s_0 t - k_c y)} + \text{c.c.}$$

Augmented by higher-order terms, this gives a pair of counter-rotating waves with respect to y. In the original coordinate system, these waves travel in both spatial directions simultaneously: $U_{\pm}(x,y,t) = U(x-\omega t, y \pm st/k)$.

The secondary waves still obey some symmetries, which may be broken in tertiary bifurcations. For instance, the quasi-stationary wave could become unstable to a time-periodic solution, while the interaction of standing and rotating travelling waves would lead to other types of quasi-periodic waves based on three incommensurable frequencies, and so on. This analysis, however, is beyond the scope of the present contribution.

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Appendix. The higher-order terms and the perturbed eigenvalues of the transverse modes

Here we evaluate the development of the perturbation modes (3.12) and (3.17), with fixed transverse wavenumber k, along the one-dimensional vertically travelling periodic wave of §2.

A.1. *Preliminaries*

In general, the expansion of §3.1 allows the calculation of the remaining physical variables from ψ_j and the inhomogeneous terms in the following way. First

$$\mathrm{i}kw_j + \lambda_0 u_j' = (\sigma_0\psi_j - \lambda_0\omega_0\psi_j' + f_\psi^j)/(1-\phi_0) \tag{A 1a}$$

follows from (3.6*a*) and, hence,

$$k^2 q_j - \lambda_0^2 q_j'' = -\frac{\sigma_0}{\varphi_0(1-\phi_0)}\psi_j + \frac{\lambda_0}{\varphi_0}\left(\frac{\omega_0}{1-\phi_0} + p_0'\varphi_0'\right)\psi_j' - \frac{f_\psi^j}{\varphi_0(1-\phi_0)} + \frac{f_q^j}{\varphi_0} \tag{A 1b}$$

follows from (3.6*b*). Then (3.6*c*) gives

$$\begin{aligned}[F\sigma_0(1-\phi_0) + \mu k^2]u_j - Fw_0\lambda_0(1-\phi_0)u_j' - \mu\lambda_0^2 u_j'' \\ = \psi_j - \lambda_0 G_0\psi_j' - \lambda_0 q_j' + \mu\kappa\lambda_0(\mathrm{i}kw_j + \lambda_0 u_j')' + f_u^j, \end{aligned} \tag{A 1c}$$

while (3.6d) leads to

$$[F\sigma_0(1-\phi_0)+\mu k^2]\,w_j - F\omega_0\lambda_0(1-\phi_0)\,w_j' - \mu\lambda_0^2 w_j'' = -\mathrm{i}k\,[G_0\psi_j+q_j-\mu\kappa(\mathrm{i}kw_j+\lambda_0 u_j')]+f_w^j. \quad (A\,1d)$$

The mean values of the dependent variables are determined by averaging these equations; averaging (3.8a) for ψ_j does not then give a new result. At $O(\epsilon)$,

$$\mathrm{K}_0\psi_1 = F^1 \quad (A\,2)$$

is obtained, where according to (3.8a) F^1 is composed of

$$f_\psi^1 = \sigma_1\psi_0+\mathrm{i}k\phi_1 w_0+\lambda_0(\phi_1 u_0+v_1\psi_0)', \quad (A\,3a)$$

$$f_q^1 = -k^2\varphi_0'\phi_1 q_0+\lambda_0^2\varphi_0'(p_1'\psi_0+\phi_1 q_0')'+\lambda_0 p_0'\varphi_0''(\phi_1\psi_0)', \quad (A\,3b)$$

$$f_u^1 = -F\sigma_1(1-\phi_0)\,u_0+F\sigma_0\phi_1 u_0-F\lambda_0[(1-\phi_0)\,u_0+\omega_0\psi_0]\,v_1'-\lambda_0 G_0'(\phi_1\psi_0)', \quad (A\,3c)$$

$$f_w^1 = -F\sigma_1(1-\phi_0)\,w_0+F\sigma_0\phi_1 w_0-\mathrm{i}kG_0'\phi_1\psi_0. \quad (A\,3d)$$

At this stage we have to determine the zeroth-order mode from which we want to start. So let us now choose the pure transverse mode (3.15) and follow it up to higher orders in ϵ. The description of the fate of the other transverse mode, the pure velocity perturbation (3.17), is much shorter and given in §A.4.

A.2. *The initially pure transverse mode*

First, complementing the considerations in §3.2, we remark that the eigenvalues σ_0 of (3.12a) are real for all values of k, if

$$\omega_0^2 < \frac{c}{F^2\varphi_0(1-\phi_0)}; \quad (A\,4a)$$

when this condition is violated, they appear as a complex-conjugate pair for $k^2\in(k_-^2, k_+^2)$, where

$$k_\pm^2 = 2\frac{F^2}{c^2}\left(\omega_0^2-\frac{c}{2F^2\phi_0(1-\phi_0)}\pm\omega_0\left[\omega_0^2-\frac{c}{F^2\varphi_0(1-\phi_0)}\right]^{1/2}\right), \quad (A\,4b)$$

and this property carries over to the perturbed eigenvalues. To begin with the formal derivations, we note that owing to (3.12a) the first part of the operator K_0 vanishes, such that it contains differential operators of first, second and third orders only. As a consequence, the solvability condition for (3.8a) is the F^j must not contain any constants. This rests on the fact that the only periodic solutions of the homogeneous adjoint equation to (3.8a) with the side condition (3.12a) are constants. Moreover, it leads to the uniqueness condition that no ψ_j except ψ_0 must contain any constants either. Thus, we have the two conditions

$$\overline{F^j}=0, \quad \overline{\psi_j}=0 \quad \text{for all} \quad j>0. \quad (A\,5)$$

Inserting the solution (3.12) into (A 2) gives

$$c\lambda_0^3\omega_0\psi_1'''-c\lambda_0^2\sigma_0\psi_1''+\lambda_0\omega_0(c\lambda_0^2-2F\sigma_0-ck^2)\,\psi_1' = \sigma_1 A_0\psi_0+A_1\phi_1+\lambda_0 A_2\phi_1', \quad (A\,6)$$

with

$$A_0 = F(\omega_0^2 k^2 - \sigma_0^2)/\sigma_0, \tag{A 7a}$$

$$A_1 = G_0'(\lambda_0^2+k^2)\,\psi_0 - k^2\varphi_0' q_0/\varphi_0 - 2F\omega_0 \lambda_0^2 u_0 - (\tilde{c}+ck^2)\,\mathrm{i}kw_0, \tag{A 7b}$$

$$A_2 = \left[\frac{1}{\varphi_0}(p_0'\varphi_0''+\alpha\varphi_0') + \frac{\omega_0}{1-\phi_0}(F\sigma_0+\tilde{c}+ck^2)\right]\psi_0 - (\tilde{c}+ck^2)\,u_0 + \mathrm{i}kF\omega_0 w_0. \tag{A 7c}$$

The solvability condition requires $\sigma_1 A_0 = 0$ to hold for arbitrary values of k, so that $\sigma_1 = 0$ is obtained. Taking the averages of equations (A 1*a–d*) for $j = 1$ leads to the following relations:

$$\mathrm{i}k(1-\phi_0)\,\overline{w_1} = \sigma_0\,\overline{\psi_1} + \sigma_1\,\overline{\psi_0} = -\varphi_0(1-\phi_0)\,k^2\overline{q_1}, \tag{A 8a}$$

$$[F\sigma_0(1-\phi_0)+\mu k^2]\,\overline{u_1} = \overline{\psi_1} - F\sigma_1(1-\phi_0)\,\overline{u_0}, \tag{A 8b}$$

$$(F\sigma_0+ck^2)(1-\phi_0)\,\overline{w_1} = -\mathrm{i}k(G_0\overline{\psi_1}+\overline{q_1}) - F\sigma_1(1-\phi_0)\,\overline{w_0}, \tag{A 8c}$$

from which again $\sigma_1 = 0$ is deduced. Incorporating the uniqueness condition $\overline{\psi_1} = 0$ then yields

$$\overline{q_1} = \overline{u_1} = \overline{w_1} = 0. \tag{A 9}$$

Now, solving (A 6) gives

$$\left.\begin{aligned} &\psi_1 = B_1\,\mathrm{e}^{\mathrm{i}z} + B_2\,\mathrm{e}^{-\mathrm{i}z},\\ &B_1 = (A_1+\mathrm{i}\lambda_0 A_2)/C_1, \quad B_2 = (A_1-\mathrm{i}\lambda_0 A_2)/C_2,\\ &C_1 = c\lambda_0^2\sigma_0 - \mathrm{i}\lambda_0\omega_0(2F\sigma_0+ck^2), \quad C_2 = C_1(\mathrm{i}\rightarrow -\mathrm{i}). \end{aligned}\right\} \tag{A 10}$$

Then (A 1*a*) yields the expression

$$\left.\begin{aligned} &\mathrm{i}kw_1 + \lambda_0 u_1' = D_1\,\mathrm{e}^{\mathrm{i}z} + D_2\,\mathrm{e}^{-\mathrm{i}z},\\ &D_1 = \frac{\sigma_0 - \mathrm{i}\lambda_0\omega_0}{1-\phi_0}B_1 + \frac{\mathrm{i}\lambda_0}{1-\phi_0}\left(u_0 - \frac{\omega_0}{1-\phi_0}\psi_0\right) + \frac{\mathrm{i}kw_0}{1-\phi_0},\\ &D_2 = D_1(B_1\rightarrow B_2, \quad \mathrm{i}\rightarrow -\mathrm{i}, \quad k\rightarrow -k). \end{aligned}\right\} \tag{A 11}$$

Because $q_1 \sim \exp(\pm\mathrm{i}z)$, (A 1*b*) gives

$$\varphi_0(k^2+\lambda_0^2)\,q_1 = \lambda_0 p_0'\varphi_0'\psi_1' - (\mathrm{i}kw_1+\lambda_0 u_1') - k^2\varphi_0'\phi_1 u_0 + \lambda_0(p_0'\varphi_0''+\alpha\varphi_0')\,\phi_0')\,\phi_1'\psi_0. \tag{A 12}$$

We shall not need u_1 in what follows but we shall need w_1:

$$\left.\begin{aligned} &w_1 = \mathrm{i}k(E_1\,\mathrm{e}^{\mathrm{i}z} + E_2\,\mathrm{e}^{-\mathrm{i}z}), \quad E_1 = F_1/G_1, \quad E_2 = F_2/G_2,\\ &F_1 = \left[F\omega_0^2 - \mathrm{i}\lambda_0\frac{p_0'\varphi_0'}{\varphi_0(\lambda_0^2+k^2)}\right]B_1 + \left[\mu\kappa + \frac{1}{\varphi_0(\lambda_0^2+k^2)}\right]D_1\\ &\qquad - \left[G_0' + \mathrm{i}\lambda_0\frac{p_0'\varphi_0''+\alpha\varphi_0'}{\varphi_0(\lambda_0^2+k^2)}\right]\psi_0 + \frac{k^2\varphi_0'}{\varphi_0(\lambda_0^2+k^2)}q_0 + \frac{F\sigma_0}{\mathrm{i}k}w_0,\\ &G_1 = F(1-\phi_0)(\sigma_0 - \mathrm{i}\omega_0\lambda_0) + \mu(\lambda_0^2+k^2),\\ &F_2 = F_1(B_1\rightarrow B_2, D_1\rightarrow D_2, \mathrm{i}\rightarrow -\mathrm{i}, k\rightarrow -k), \quad G_2 = G_1(\mathrm{i}\rightarrow -\mathrm{i}). \end{aligned}\right\} \tag{A 13}$$

At $O(\epsilon^2)$ we need only the averaged equations, namely

$$\mathrm{i}k(1-\phi_0)\overline{w_2} = \sigma_2\overline{\psi_0}+\mathrm{i}k(\overline{\phi_1 w_1+\phi_2 w_0}), \tag{A 14a}$$

$$\mathrm{i}k\overline{w_2}+k^2\varphi_0\overline{q_2} = -k^2\left[\varphi_0'(\overline{\phi_1 q_1}+\overline{\phi_2 q_0})+\frac{\varphi_0''}{2}\overline{\phi_1^2 q_0}\right], \tag{A 14b}$$

$$\begin{aligned}[F\sigma_0(1-\phi_0)+\mu k^2]\overline{u_2}+\lambda_0\overline{q_2'} &= -F\sigma_2(1-\phi_0)\overline{u_0}+F\sigma_0(\overline{\phi_1 u_1}+\overline{\phi_2 u_0})\\ &\quad -F\lambda_0\overline{[(1-\phi_0)u_0+\omega_0\psi_0]v_2'}-F\lambda_0\overline{[(1-\phi_0)u_1+\omega_0\psi_1-\phi_1 u_0-v_1\psi_0]v_1'},\end{aligned} \tag{A 14c}$$

$$\begin{aligned}(F\sigma_0+ck^2)(1-\phi_0)\overline{w_2}+\mathrm{i}k\overline{q_2} &= -F\sigma_2(1-\phi_0)\overline{w_0}\\ &\quad +F\sigma_0(\overline{\phi_1 w_1}+\overline{\phi_2 w_0})-\mathrm{i}k\,[G_0'(\overline{\phi_1\psi_1}+\overline{\phi_2\psi_0})+\tfrac{1}{2}G_0''\overline{\phi_1^2\psi_0}],\end{aligned} \tag{A 14d}$$

from which we derive an expression for σ_2:

$$\begin{aligned}-A_0\psi_0\sigma_2 &= (2G_0'\varphi^c+G_0'')k^2\psi_0+\frac{k^2}{\varphi_0}\left(\frac{2k^2\varphi_0'^2}{\varphi_0(\lambda_0^2+k^2)}-2\varphi_0'\varphi^c-\varphi_0''\right)q_0\\ &\quad+\left(G_0'+\frac{\varphi_0'\sigma_0}{(1-\phi_0)\varphi_0^2(\lambda_0^2+k^2)}\right)k^2(B_1+B_2)\\ &\quad+\mathrm{i}\lambda_0\frac{k^2\alpha\varphi_0'}{\varphi_0(\lambda_0^2+k^2)}(B_1-B_2)+k^2\left(ck^2+\frac{1}{\varphi_0(1-\phi_0)}\right)(E_1+E_2)\\ &\quad+2\mathrm{i}kw_0\left[\frac{k^2\varphi_0'}{(1-\phi_0)\varphi_0^2(\lambda_0^2+k^2)}-\left(ck^2+\frac{1}{\varphi_0(1-\phi_0)}\right)\varphi^c\right].\end{aligned} \tag{A 15}$$

As this relation does not seem applicable to a general analysis, its long- and short-wave behaviour will be examined. We only mention that in sample numerical evaluations of (A 15) it was always found that σ_2 is negative for sufficiently large values of k^2, while on the other hand it assumed positive as well as negative values for small k^2.

A.3. *Asymptotic considerations*

We evaluate formula (A 15) for small and large k^2. It is kept in mind that in these limits σ_0 is real, hence the eigenfunction U_{+k} is also real (except for the transverse velocity which is imaginary) and the coefficients $B_2, \ldots, G_2$ are complex conjugate to $B_1, \ldots, G_1$.

A.3.1. *The long-wave limit*

An instability seems most likely to occur for small k^2, since (3.14) states that $\sigma_{0,1} \sim -k^2 \to 0^-$. In this regime, the terms proportional to $u_0 = O(k^{-2})\,\psi_0$ give the dominant contribution, as can be seen from (3.12*b*). In addition,

$$\mathrm{i}kw_0 = O(k^2)\,\psi_0, \quad q_0 = O(1)\,\psi_0; \quad -A_0 = \frac{1}{\varphi_0(1-\phi_0)}+O(k^2).$$

Thus,

$$A_1 = -2F\omega_0\lambda_0^2 u_0+O(1)\,\psi_0, \quad A_2 = -\tilde{c}u_0+O(1)\,\psi_0,$$

$$k^2B_1 = -\frac{k^2}{C_1}(2F\omega_0\lambda_0^2+\mathrm{i}\lambda_0\tilde{c})\,u_0+O(1)\,\psi_0,$$

where $C_1 = -k^2[\xi_1 c\lambda_0^2+\mathrm{i}\lambda_0\omega_0(c-2F\xi_1)]+O(k^4)$. Therefore,

$$B_1 = O(k^{-2})\,u_0+O(k^{-2})\,\psi_0,$$

so that the terms $\sim u_0$ can be neglected with respect to those $\sim B_1$. Furthermore,

$$(1-\phi_0)\,D_1 = \mathrm{i}\lambda_0(u_0-\omega_0 B_1)+O(k^2)\,B_1+O(1)\,\psi_0,$$

hence,

$$F_1 = \left[F\omega_0^2 - \frac{\mathrm{i}}{\lambda_0}\left(\frac{\mu\kappa\lambda_0^2\omega_0}{1-\phi_0} - \alpha\right)\right]B_1 + O(u_0) + O(k^2)B_1 + O(k^2)u_0 + O(1)\psi_0,$$

while $G_1 = \mu\lambda_0^2 - \mathrm{i}F\omega_0\lambda_0(1-\phi_0) + O(k^2)$. Putting all this together and making use of the abbreviations defined in (3.13), (3.15), and (4.13), gives

$$(1+O(k^2))\,|c\lambda_0\xi_1 + \mathrm{i}\omega_0(c-2F\xi_1)|^2\frac{\psi_0}{2u_0}\sigma_2$$

$$= \xi_4\omega_0(c\tilde{c} - 2F^2\omega_0^2) + \lambda_0\xi_1\xi_5\left[c\left(\frac{1}{\varphi_0(1-\phi_0)} - c\lambda_0^2\right) - 4F^2\omega_0^2\right] + O(1)\psi_0. \quad \text{(A 16)}$$

In addition,

$$\psi_0/u_0 = [\mu - F(1-\phi_0)\xi_1 + O(k^2)]k^2. \quad \text{(A 17)}$$

We are interested mainly in the sign of σ_2, but since it depends on the various parameters in a complicated manner, no general statement appears to be achievable; in particular, the individual terms in (A 16) show opposite behaviour. Note that the sign of σ_2 depends on the difference of the leading-order eigenvalues of the two transverse modes.

A.3.2. *The short-wave limit*

Considering large k^2 we see that $\sigma_{0,1} = O(1)$ and thus $u_0 = O(k^{-2})$, $q_0 = O(k^{-2})$, $\mathrm{i}kw_0 = O(1)$, whence $C_{1,2}$ and A_0 are $O(k^2)$. Moreover, $A_{1,2} = O(k^2)$, $-k^2/A_0 \to 1/c$, $(B_1 - B_2)/k^2 \to 0$. Hence,

$$c\sigma_2 \to 2G_0'\varphi^c + G_0'' + G_0' \lim_{k^2\to\infty}[B_1 + B_2 + ck^2(E_1+E_2)] + 2F\omega_0^2\varphi^c/(1-\phi_0).$$

Without going into further details, we state that

$$c\lim_{k^2\to\infty}\sigma_2 = G_0'' + 2G_0'\left(\varphi^c - \frac{2}{1-\phi_0}\right) - \frac{2G_0}{1-\phi_0}\left(\varphi^c - \frac{1}{1-\phi_0}\right), \quad \text{(A 18)}$$

showing that in this limit the perturbed eigenvalue tends towards a constant, which above all depends on the interparticle pressure.

A.4. *The pure velocity perturbation mode*

The evaluation of the fate of the initially pure velocity perturbation mode (3.17) is very similar. Because here $\psi_0 = q_0 = w_0 = 0$, $u_0 = \text{const.} \neq 0$, the inhomogeneous terms (A 3) simplify greatly:

$$f_\psi^1 = \lambda_0 u_0\phi_1', \quad f_q^1 = f_w^1 = 0, \quad f_u^1 = F\sigma_0 u_0\phi_1 + F\lambda_0\omega_0 u_0\phi_1'. \quad \text{(A 19)}$$

On the other hand, we have to deal with the full operator K_0, because none of its coefficients vanishes. But then the homogeneous adjoint equation $\mathrm{K}_0^*\psi^* = 0$ has no non-trivial periodic solutions, so that no solvability condition has to be satisfied. The ψ_j may now assume mean values determined by

$$\alpha_1\overline{\psi_j} = \overline{F^j}, \quad \alpha_1 = F\sigma_0^2 + \sigma_0\left(ck^2 + \frac{1}{\varphi_0(1-\phi_0)}\right) + F\omega_0^2k^2, \quad \text{(A 20)}$$

where σ_0 is given by (3.17). At $O(\epsilon)$ one obtains 2π-periodic solutions with $\overline{\psi_1} = \overline{q_1} = \overline{w_1} = 0$, and $\sigma_1 = 0$:

$$(\psi, q, u, w)_1 = (0, 0, \overline{u_1}, 0) + u_0[(\psi, q, u, \mathrm{i}kw)_+\mathrm{e}^{\mathrm{i}z} + (\psi, q, u, \mathrm{i}kw)_-\mathrm{e}^{-\mathrm{i}z}], \quad \text{(A 21)}$$

where the quantities with the minus index are complex conjugates of those with the plus index. The latter follow from the relations

$$[\alpha_1+\sigma_0 c\lambda_0^2-\mathrm{i}\lambda_0\omega_0(2F\sigma_0+ck^2)]\psi_+ = -2F\omega_0\lambda_0^2 + \mathrm{i}\lambda_0\left[2F\sigma_0+ck^2+\frac{1}{\varphi_0(1-\phi_0)}-c\lambda_0^2\right], \quad \text{(A 22a)}$$

$$(k^2+\lambda_0^2)q_+ = -\left[\frac{\sigma_0}{\varphi_0(1-\phi_0)}+\mathrm{i}\lambda_0(1+c\lambda_0^2\omega_0)\right]\psi_+ - \frac{\mathrm{i}\lambda_0}{\varphi_0(1-\phi_0)}, \quad \text{(A 22b)}$$

$$[\mu\lambda_0^2-\mathrm{i}F\omega_0\lambda_0(1-\phi_0)]u_+ = \left[1+\frac{\mu\kappa\lambda_0^2\omega_0}{1-\phi_0}+\mathrm{i}\lambda_0\left(F\omega_0^2+\frac{\mu\kappa\sigma_0}{1-\phi_0}\right)\right]\psi_+ - \mathrm{i}\lambda_0 q_+ - \frac{\mu\kappa\lambda_0^2}{1-\phi_0}+F\sigma_0+\mathrm{i}F\omega_0\lambda_0, \quad \text{(A 22c)}$$

$$[\mu\lambda_0^2-\mathrm{i}F\omega_0\lambda_0(1-\phi_0)]w_+ = \left(F\omega_0^2+\frac{\mu\kappa\sigma_0}{1-\phi_0}-\mathrm{i}\frac{\mu\kappa\lambda_0\omega_0}{1-\phi_0}\right)\psi_+ - q_+ - \frac{\mathrm{i}\lambda_0}{1-\phi_0}. \quad \text{(A 22d)}$$

The value of $\overline{u_1}$ remains undetermined but is also irrelevant up to order ϵ^2. Again one needs only the average expressions at $O(\epsilon^2)$, which now read

$$-\sigma_0\overline{\psi_2}+\mathrm{i}k(1-\phi_0)\overline{w_2} = \mathrm{i}k\overline{\phi_1 w_1},$$
$$\mathrm{i}k\overline{w_2}+k^2\varphi_0\overline{q_2} = -k^2\varphi_0'\overline{\phi_1 q_1},$$
$$\overline{\psi_2} = F(1-\phi_0)\sigma_2 u_0 - F\sigma_0\overline{\phi_1 u_1} - F\sigma_0 u_0\overline{\phi_2} - F\lambda_0\omega_0\left(\overline{\phi_1' u_1}+\frac{\omega_0}{1-\phi_0}\overline{\phi_1'\psi_1}\right),$$
$$(F\sigma_0+ck^2)(1-\phi_0)\overline{w_2}+\mathrm{i}k(\overline{q_2}+G_0\overline{\psi_2}) = F\sigma_0\overline{\phi_1 w_1}-\mathrm{i}kG_0'\overline{\phi_1\psi_1}.$$

These equations yield the perturbed eigenvalue as

$$\left.\begin{aligned}
\frac{u_0}{1-\phi_0}[\ldots]\sigma_2 &= \frac{\overline{\phi_1 w_1}}{\mathrm{i}k}\left(ck^2+\frac{1}{\varphi_0(1-\phi_0)}\right)-\frac{\varphi_0'}{\varphi_0}\overline{\phi_1 q_1}+G_0'\overline{\phi_1\psi_1} \\
&\quad +\frac{1}{(1-\phi_0)^2}[\ldots]\left(\sigma_0\overline{\phi_1 u_1}+\sigma_0 u_0\overline{\phi_2}-\frac{\sigma_0\omega_0}{1-\phi_0}\overline{\phi_1\psi_1}+\mathrm{i}k\omega_0\overline{\phi_1 w_1}\right), \\
[\ldots] &= F(1-\phi_0)\xi_1-\mu\left(\mu\kappa k^2+\frac{1}{\varphi_0}\right),
\end{aligned}\right\} \quad \text{(A 23)}$$

which behaves in the long-wave limit like

$$\left(\xi_1-\frac{\mu}{F(1-\phi_0)}\right)\sigma_2 = \frac{2}{k^2}\mathrm{Re}[\lambda_0(\xi_4+\mathrm{i}\xi_5)R]+O(1), \quad k^2 \ll 1, \quad \text{(A 24c)}$$

$$R = \frac{-2F\omega_0\lambda_0+\mathrm{i}\{1/[(\varphi_0(1-\phi_0)]-c\lambda_0^2\}}{F^2\omega_0^2(1-\phi_0)-\mu\tilde{c}+\mathrm{i}F\lambda_0\omega_0\mu(1-\kappa)}. \quad \text{(A 24b)}$$

Note again that the sign of this σ_2 depends on the difference of the leading-order eigenvalues of both transverse modes; the larger one plays the role described at the end of §4 (cf. table 1). The short-wave behaviour is given by

$$\sigma_2 = \frac{2\mu^3}{F(1-\phi_0)^3|\mu\lambda_0-\mathrm{i}F\omega_0(1-\phi_0)|^2}k^4+O(k^2), \quad k^2 \gg 1. \quad \text{(A 25)}$$

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